One Concrete Application of Bernstein Sets in Measure Theory

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Paradoxical Subsets of $\mathbf{R}$

- Vitali Set
- Hamel Bases
- Bernstein Set
- Luzini Set
- Sierpinski Set

Bernstein Sets

Definition

Let X be a subset of the real line $\mathbf{R}$. We say that X is a Bernstein set in $\mathbf{R}$ if, for every non-empty perfect set $P \subset \mathbf{R}$, both intersections

$$P \cap X \text{ and } P \cap (\mathbf{R} \setminus X)$$

are nonempty.

Relationships between some Paradoxical Sets

Theorem

There exists a subset X of $\mathbf{R}$ such that X is simultaneously a Vitali set and a Bernstein set.

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Theorem

There exists a Hamel basis of $\mathbf{R}$ which simultaneously is a Bernstein set.

General Measure Extension Problem

Problem of measure extension has a three aspects:

- Pure set-theoretical
- Algebraic
- Topological

The Role of Bernstein Sets

- In particular, we envisage Bernstein subsets of the real line **R** from the point of view of their measurability with respect to certain classes of measures on **R**.
- The importance of Bernstein sets in various questions of general topology, measure theory an the theory of Boolean algebras is well known.

Some Auxiliary Notions

Definition

- We shall say that a set $X \subset E$ is **absolutely measurable** with respect to $\mathcal{M}$ if, for an arbitrary measure $\mu \in \mathcal{M}$, the set X is measurable with respect to μ.

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- We shall say that a set $Y \subset E$ is **relatively measurable** with respect to the class $\mathcal{M}$ if there exists at least one measure $\mu \in \mathcal{M}$ such that Y is measurable with respect to μ.

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- We shall say that a set $Z \subset E$ is **absolutely nonmeasurable** with respect to $\mathcal{M}$ if there exists no measure $\mu \in \mathcal{M}$ such that Z is measurable with respect to μ.

Lemma

There exists a Bernstein set $X \subset \mathbf{R}$ almost invariant with respect to the group of all translations of $\mathbf{R}$, i.e., the relation

$$(\forall h \in \mathbf{R})(\mathrm{card}((h + X)\triangle X) < \mathbf{c})$$

holds true.

Theorem

There is a Bernstein set X in $\mathbf{R}$ such that:
(1) there exists a translation invariant measure μ_1 on $\mathbf{R}$ extending the Lebesgue measure λ and satisfying the equality $\mu_1(X) = 0$;
(2) there exists a translation invariant measure μ_2 on $\mathbf{R}$ extending the Lebesgue measure λ and satisfying the equality $\mu_2(\mathbf{R} \setminus X) = 0$

Some Auxiliary Notions

Definition

Let **E** be a base (ground) set and let μ be a measure defined on some σ-algebra of subset of **E**.

- Recall that μ is said to be diffused (or continuous) if all singletons in **E** belong to the domain of μ and μ vanishes at all of them

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- A set $Z \subset E$ is said to be μ-thick, if for every μ-measurable set $X \subset E$ with $\lambda(X) > 0$, we have $Z \cap X \neq \emptyset$.

Example

Let M denote the class of the completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$. It is not difficult to see that if B is any Bernstein set in $\mathbf{R}$ and μ is any measure from the class M, then both B and $\mathbf{R} \setminus B$ are μ-thick subsets of $\mathbf{R}$ and, consequently, they are nonmeasurable with respect to μ.

Description of Bernstein Sets

More precisely, for a subset T of $\mathbf{R}$, the following two assertions are equivalent:

1. T is a Bernstein set;
2. for every measure $\mu \in M$, the set T is nonmeasurable with respect to μ;

In, particular, assertion (2) indicates that all Bernstein sets have extremely bad properties from the point of view of topological measure theory.

Some Auxilary Notions

Definition

Let $(G, +)$ be a commutative group and let μ be a nonzero σ-finite measure defined on some σ-algebra of subsets of G Let H be a subgroup of G.

- Recall that μ is an H-quasi-invariant measure if the domain of μ and σ -ideal generated by all μ-measure zero sets are H-invariant classes of subsets of G.

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- Recall that μ is an H-invariant measure if the dom(μ) is an H-invariant class of subsets of G and the equality $\mu(h + X) = \mu(X)$ is satisfied for any element $h \in H$ and any set $X \in dom(\mu)$.

Absolutely Negligible Sets

Definition

A set $Z \subset X$ is called H-absolutely negligible, if for every measure $\mu \in M(G,H)$ there exists a measure $\mu' \in M(G,H)$ extending μ and such that $\mu'(Z) = 0$.

Example

Hamel Bases are H-Absolutely Negligible Sets.

Absolutely Negligible Sets

Theorem

There exists a Bernstein set which is absolutely negligible with respect to the class of all nonzero σ-finite translation invariant measures on $\mathbf{R}$

More General

Let G coincide with the additive group $\mathbf{R}$ and let $H \subset \mathbf{R}$ be an uncountable vector space over the field $\mathbf{Q}$ of all rational numbers. We are going to demonstrate that:

- there exists a Bernstein set B on $\mathbf{R}$ which is absolutely nonmeasurable with respect to the class $M(\mathbf{R}, H)$

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Let G coincide with the additive group $\mathbf{R}$ and let $H \subset \mathbf{R}$ be an uncountable vector space over the field $\mathbf{Q}$ of all rational numbers. We are going to demonstrate that:

- there exists a Bernstein set B on $\mathbf{R}$ which is absolutely nonmeasurable with respect to the class $M(\mathbf{R}, H)$
- there exists a countable family $\{B_i : i \in I\}$ of Bernstein subsets of $\mathbf{R}$ such that $\cup\{B_i : i \in I\}$ and each $B_i, i \in I$ is an H-absolutely negligible set.

Thank You for Your Attention