Some structural properties of ideal invariant injections

Jarosław Swaczyna

Łódź University of Technology

Hejnice, 05.02.2016

joint work with Marek Balcerzak and Szymon Głąb

Introduction

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in$ **Inj**. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in$ **Inj** is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact

Let $\mathcal{I}$ be an ideal on ω and let $f \in$ **Inj**.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.

(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in$ **Inj**. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in$ **Inj** is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact

Let $\mathcal{I}$ be an ideal on ω and let $f \in$ **Inj**.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.

(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.

Introduction

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in$ **Inj**. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in$ **Inj** is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact

Let $\mathcal{I}$ be an ideal on ω and let $f \in$ **Inj**.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.

(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.

Introduction

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in \textbf{Inj}$. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in \textbf{Inj}$ is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact

Let $\mathcal{I}$ be an ideal on ω and let $f \in \textbf{Inj}$.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.

(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.

Introduction

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in \mathbf{Inj}$. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in \mathbf{Inj}$ is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact

Let $\mathcal{I}$ be an ideal on ω and let $f \in \mathbf{Inj}$.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.

(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in$ **Inj**. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in$ **Inj** is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact

Let $\mathcal{I}$ be an ideal on ω and let $f \in$ **Inj**.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.

(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.

We will work with injections from ω to ω. The set of all such injections will be denoted by **Inj**. Fix an ideal $\mathcal{I}$ on ω and let $f \in \textbf{Inj}$. We say that f is *$\mathcal{I}$-invariant* if $f[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. We say that f^{-1} is *$\mathcal{I}$-invariant* if $f^{-1}[A] \in \mathcal{I}$ for all $A \in \mathcal{I}$. If f and f^{-1} are $\mathcal{I}$-invariant then f is called *bi-$\mathcal{I}$-invariant*. Note that every $f \in \textbf{Inj}$ is bi-Fin-invariant.

We start from easy facts and simple examples.

Fact

Let $\mathcal{I}$ be an ideal on ω and let $f \in \textbf{Inj}$.

(i) f^{-1} is $\mathcal{I}$-invariant if and only if $f[A] \notin \mathcal{I}$ for every $A \notin \mathcal{I}$.

(ii) If $f[\omega] \in \mathcal{I}$, then f is $\mathcal{I}$-invariant and it is not bi-$\mathcal{I}$-invariant.

(iii) If $\mathrm{Fix}(f) \in \mathcal{I}^\star$, then f is bi-$\mathcal{I}$-invariant.

(iv) **Inj** is a G_δ subset of ω^ω, hence it is a Polish space.

Easy examples

(i) Note that every increasing injection is $\mathcal{I}_d$-invariant. In particular, $f(n) := n^2$ is $\mathcal{I}_d$-invariant. Moreover, in this case $f[\omega] \in \mathcal{I}_d$, hence f is not bi-$\mathcal{I}_d$-invariant.

(ii) Let $f\colon \omega \to \omega$ be given by the formulas: $f(2n) := 4n$, $f(4n+1) = 4n+2$, $f(4n+3) := 2n+1$ for $n \in \omega$. Then f is a bijection. Consider the ideal $\mathcal{I}$ defined as follows

$$\mathcal{I} := \{A \cup B\colon A \in \mathrm{Fin},\ B \subseteq 2\omega\}.$$

Clearly, f is $\mathcal{I}$-invariant bijection which is not bi-$\mathcal{I}$-invariant.

Easy examples

(i) Note that every increasing injection is $\mathcal{I}_d$-invariant. In particular, $f(n) := n^2$ is $\mathcal{I}_d$-invariant. Moreover, in this case $f[\omega] \in \mathcal{I}_d$, hence f is not bi-$\mathcal{I}_d$-invariant.

(ii) Let $f\colon \omega \to \omega$ be given by the formulas: $f(2n) := 4n$, $f(4n+1) = 4n+2$, $f(4n+3) := 2n+1$ for $n \in \omega$. Then f is a bijection. Consider the ideal $\mathcal{I}$ defined as follows

$$\mathcal{I} := \{A \cup B\colon\ A \in \mathrm{Fin},\ B \subseteq 2\omega\}.$$

Clearly, f is $\mathcal{I}$-invariant bijection which is not bi-$\mathcal{I}$-invariant.

Countably generated ideals

Fact

There are three types of countably generated ideals: Fin, $\mathrm{Fin} \oplus \mathcal{P}(\omega)$ and $\mathrm{Fin} \times \emptyset$.

Theorem

- Each $f \in \mathbf{Inj}$ is bi-Fin-invariant.
- Sets $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, and **bi**-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all bi-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, are true F_σ subsets of **Inj**.
- The sets $\mathrm{Fin} \times \emptyset$-**Inv** and **bi**-$\mathrm{Fin} \times \emptyset$-**Inv**, are meager of type $F_{\sigma\delta}$ in $\mathbf{Inj} \subseteq (\omega \times \omega)^{\omega \times \omega}$. Moreover, **bi**-$\mathcal{I}$-**Inv** is $F_{\sigma\delta}$-complete.

Countably generated ideals

Fact

There are three types of countably generated ideals: Fin, $\mathrm{Fin} \oplus \mathcal{P}(\omega)$ and $\mathrm{Fin} \times \emptyset$.

Theorem

- Each $f \in \mathbf{Inj}$ is bi-Fin-invariant.
- Sets $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, and **bi**-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all bi-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, are true F_σ subsets of **Inj**.
- The sets $\mathrm{Fin} \times \emptyset$-**Inv** and **bi**-$\mathrm{Fin} \times \emptyset$-**Inv**, are meager of type $F_{\sigma\delta}$ in $\mathbf{Inj} \subseteq (\omega \times \omega)^{\omega \times \omega}$. Moreover, **bi**-$\mathcal{I}$-**Inv** is $F_{\sigma\delta}$-complete.

Countably generated ideals

Fact

There are three types of countably generated ideals: Fin, $\mathrm{Fin} \oplus \mathcal{P}(\omega)$ and $\mathrm{Fin} \times \emptyset$.

Theorem

- Each $f \in \mathbf{Inj}$ is bi-Fin-invariant.
- Sets $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, and **bi**-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all bi-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, are true F_σ subsets of **Inj**.
- The sets $\mathrm{Fin} \times \emptyset$-**Inv** and **bi**-$\mathrm{Fin} \times \emptyset$-**Inv**, are meager of type $F_{\sigma\delta}$ in $\mathbf{Inj} \subseteq (\omega \times \omega)^{\omega \times \omega}$. Moreover, **bi**-$\mathcal{I}$-**Inv** is $F_{\sigma\delta}$-complete.

Countably generated ideals

Fact

There are three types of countably generated ideals: Fin, $\mathrm{Fin} \oplus \mathcal{P}(\omega)$ and $\mathrm{Fin} \times \emptyset$.

Theorem

- Each $f \in \mathbf{Inj}$ is bi-Fin-invariant.
- Sets $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all $\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, and **bi**-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-**Inv**, of all bi-$\mathrm{Fin} \oplus \mathcal{P}(\omega)$-invariant injections, are true F_σ subsets of **Inj**.
- The sets $\mathrm{Fin} \times \emptyset$-**Inv** and **bi**-$\mathrm{Fin} \times \emptyset$-**Inv**, are meager of type $F_{\sigma\delta}$ in $\mathbf{Inj} \subseteq (\omega \times \omega)^{\omega \times \omega}$. Moreover, **bi**-$\mathcal{I}$-**Inv** is $F_{\sigma\delta}$-complete.

Maximal ideals

$\mathcal{I}$-invariance

Let $\mathcal{I}$ be a maximal ideal. Then $f \in$ **Inj** is $\mathcal{I}$-invariant iff $f[\omega] \in \mathcal{I}$ or $Fix(f) \in \mathcal{I}^\star$.

EASY PROOF (hint: Orbit $O_f(n) := \{f^k(n) : k \in \mathbb{Z}\}$).

Corollary

Let $\mathcal{I}$ be a maximal ideal on ω and $f \in$ **Inj**. Then f is $\mathcal{I}$-invariant if and only if either $\mathrm{Fix}(f) \in \mathcal{I}^\star$ or $f[\omega] \in \mathcal{I}$.

Example

Let $\mathcal{I}, \mathcal{J}$ be non-isomorphic maximal ideals on ω and $f \in$ **Inj**. Then f is bi-$\mathcal{I} \oplus \mathcal{J}$-invariant iff $Fix(f) \in (\mathcal{I} \oplus \mathcal{J})^\star$.

Maximal ideals

$\mathcal{I}$-invariance

Let $\mathcal{I}$ be a maximal ideal. Then $f \in \textbf{Inj}$ is $\mathcal{I}$-invariant iff $f[\omega] \in \mathcal{I}$ or $Fix(f) \in \mathcal{I}^\star$.
EASY PROOF (hint: Orbit $O_f(n) := \{f^k(n) : k \in \mathbb{Z}\}$).

Corollary

Let $\mathcal{I}$ be a maximal ideal on ω and $f \in \textbf{Inj}$. Then f is $\mathcal{I}$-invariant if and only if either $\mathrm{Fix}(f) \in \mathcal{I}^\star$ or $f[\omega] \in \mathcal{I}$.

Example

Let $\mathcal{I}, \mathcal{J}$ be non-isomorphic maximal ideals on ω and $f \in \textbf{Inj}$. Then f is bi-$\mathcal{I} \oplus \mathcal{J}$-invariant iff $Fix(f) \in (\mathcal{I} \oplus \mathcal{J})^\star$.

Maximal ideals

$\mathcal{I}$-invariance

Let $\mathcal{I}$ be a maximal ideal. Then $f \in \mathbf{Inj}$ is $\mathcal{I}$-invariant iff $f[\omega] \in \mathcal{I}$ or $Fix(f) \in \mathcal{I}^\star$.
EASY PROOF (hint: Orbit $O_f(n) := \{f^k(n) : k \in \mathbb{Z}\}$).

Corollary

Let $\mathcal{I}$ be a maximal ideal on ω and $f \in \mathbf{Inj}$. Then f is $\mathcal{I}$-invariant if and only if either $\mathrm{Fix}(f) \in \mathcal{I}^\star$ or $f[\omega] \in \mathcal{I}$.

Example

Let $\mathcal{I}, \mathcal{J}$ be non-isomorphic maximal ideals on ω and $f \in \mathbf{Inj}$. Then f is bi-$\mathcal{I} \oplus \mathcal{J}$-invariant iff $Fix(f) \in (\mathcal{I} \oplus \mathcal{J})^\star$.

Maximal ideals

$\mathcal{I}$-invariance

Let $\mathcal{I}$ be a maximal ideal. Then $f \in$ **Inj** is $\mathcal{I}$-invariant iff $f[\omega] \in \mathcal{I}$ or $Fix(f) \in \mathcal{I}^\star$.
EASY PROOF (hint: Orbit $O_f(n) := \{f^k(n) : k \in \mathbb{Z}\}$).

Corollary

Let $\mathcal{I}$ be a maximal ideal on ω and $f \in$ **Inj**. Then f is $\mathcal{I}$-invariant if and only if either $\text{Fix}(f) \in \mathcal{I}^\star$ or $f[\omega] \in \mathcal{I}$.

Example

Let $\mathcal{I}, \mathcal{J}$ be non-isomorphic maximal ideals on ω and $f \in$ **Inj**. Then f is bi-$\mathcal{I} \oplus \mathcal{J}$-invariant iff $Fix(f) \in (\mathcal{I} \oplus \mathcal{J})^\star$.

Ideals generated by Solecki's submeasures

A submeasure on ω is a function $\varphi\colon \mathcal{P}(\omega) \to [0, \infty]$ such that:

- $\varphi(\emptyset) = 0$;
- if $A \subset B$ then $\varphi(A) \leq \varphi(B)$,
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,
- $\varphi(\{n\}) < \infty$ for all $n \in \omega$.

A submeasure φ is called a lower semicontinuos submeasure (in short, lscsm) if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for all $A \subset \omega$. For any lscsm φ, we consider two ideals given by

$$Exh(\varphi) = \{A \subset \omega\colon \lim_{n\to\infty} \varphi(A \setminus n) = 0\}.$$

$$Fin(\varphi) = \{A \subset \omega\colon \varphi(A) < \infty\}.$$

Ideals generated by Solecki's submeasures

A submeasure on ω is a function $\varphi\colon \mathcal{P}(\omega) \to [0, \infty]$ such that:

- $\varphi(\emptyset) = 0$;
- if $A \subset B$ then $\varphi(A) \leq \varphi(B)$,
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,
- $\varphi(\{n\}) < \infty$ for all $n \in \omega$.

A submeasure φ is called a lower semicontinuos submeasure (in short, lscsm) if $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap n)$ for all $A \subset \omega$. For any lscsm φ, we consider two ideals given by

$$Exh(\varphi) = \{A \subset \omega\colon \lim_{n\to\infty} \varphi(A \setminus n) = 0\}.$$

$$Fin(\varphi) = \{A \subset \omega\colon \varphi(A) < \infty\}.$$

Let φ be a lscsm. Then $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, $Fin(\varphi)$ is an F_σ ideal and $Exh(\varphi) \subset Fin(\varphi)$.

Theorem [Mazur, Solecki]

Let $\mathcal{I}$ be an ideal on ω. Then

- $\mathcal{I}$ is an F_σ ideal if and only if $\mathcal{I} = Fin(\varphi)$ for some lscsm φ.
- $\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = Exh(\varphi)$ for some lscsm φ.
- $\mathcal{I}$ is an F_σ P-ideal if and only if $\mathcal{I} = Fin(\varphi) = Exh(\varphi)$ for some lscsm φ.

If $\mathcal{I}$ is ideal on ω then it is not a G_δ set.

Ideals generated by Solecki's submeasures

Let φ be a lscsm. Then $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal, $Fin(\varphi)$ is an F_σ ideal and $Exh(\varphi) \subset Fin(\varphi)$.

Theorem [Mazur, Solecki]

Let $\mathcal{I}$ be an ideal on ω. Then

- $\mathcal{I}$ is an F_σ ideal if and only if $\mathcal{I} = Fin(\varphi)$ for some lscsm φ.
- $\mathcal{I}$ is an analytic P-ideal if and only if $\mathcal{I} = Exh(\varphi)$ for some lscsm φ.
- $\mathcal{I}$ is an F_σ P-ideal if and only if $\mathcal{I} = Fin(\varphi) = Exh(\varphi)$ for some lscsm φ.

If $\mathcal{I}$ is ideal on ω then it is not a G_δ set.

Ideals generated by Solecki's submeasures

Theorem

Let φ be a lsc submeasure on ω. Let $f\colon \omega \to \omega$ be an increasing injection and $C_f > 0$ be a constant depending on f such that $\varphi(A) \geq C_f \varphi(f[A])$ for every $A \subseteq \omega$. Then f is invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$. Additionally, if there is a constant $C'_f > 0$ with $\varphi(A) \geq C'_f \varphi(f^{-1}[A])$ for every $A \subseteq \omega$, then f is bi-invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$.

Remark

by the lower semicontinuity of φ, one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds only for finite sets $A \subseteq \omega$. It is natural to ask whether one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds for any A with $|A| \leq n$ for some fixed n. The answer is "no".

Ideals generated by Solecki's submeasures

Theorem

Let φ be a lsc submeasure on ω. Let $f\colon \omega \to \omega$ be an increasing injection and $C_f > 0$ be a constant depending on f such that $\varphi(A) \geq C_f \varphi(f[A])$ for every $A \subseteq \omega$. Then f is invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$. Additionally, if there is a constant $C'_f > 0$ with $\varphi(A) \geq C'_f \varphi(f^{-1}[A])$ for every $A \subseteq \omega$, then f is bi-invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$.

Remark

by the lower semicontinuity of φ, one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds only for finite sets $A \subseteq \omega$. It is natural to ask whether one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds for any A with $|A| \leq n$ for some fixed n. The answer is "no".

Ideals generated by Solecki's submeasures

Theorem

Let φ be a lsc submeasure on ω. Let $f\colon \omega \to \omega$ be an increasing injection and $C_f > 0$ be a constant depending on f such that $\varphi(A) \geq C_f \varphi(f[A])$ for every $A \subseteq \omega$. Then f is invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$. Additionally, if there is a constant $C'_f > 0$ with $\varphi(A) \geq C'_f \varphi(f^{-1}[A])$ for every $A \subseteq \omega$, then f is bi-invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$.

Remark

by the lower semicontinuity of φ, one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds only for finite sets $A \subseteq \omega$. It is natural to ask whether one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds for any A with $|A| \leq n$ for some fixed n.

The answer is "no".

Ideals generated by Solecki's submeasures

Theorem

Let φ be a lsc submeasure on ω. Let $f\colon \omega \to \omega$ be an increasing injection and $C_f > 0$ be a constant depending on f such that $\varphi(A) \geq C_f \varphi(f[A])$ for every $A \subseteq \omega$. Then f is invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$. Additionally, if there is a constant $C'_f > 0$ with $\varphi(A) \geq C'_f \varphi(f^{-1}[A])$ for every $A \subseteq \omega$, then f is bi-invariant with respect to the ideals $\mathrm{Fin}(\varphi)$ and $\mathrm{Exh}(\varphi)$.

Remark

by the lower semicontinuity of φ, one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds only for finite sets $A \subseteq \omega$. It is natural to ask whether one can assume that the condition $\varphi(A) \geq C_f \varphi(f[A])$ holds for any A with $|A| \leq n$ for some fixed n. The answer is "no".

Ideals $\mathcal{I}_d$ and $\mathcal{I}_{(1/n)}$

Classical density ideal

$$\mathcal{I}_d := \{A \subset \omega : \frac{card(A \cap n)}{n} \to 0\}$$

Classical summable ideal

$$\mathcal{I}_{\mathcal{S}} := \{A \subset \omega : \sum_{n \in A} \frac{1}{n} < \infty\}$$

Ideals $\mathcal{I}_d$ and $\mathcal{I}_{(1/n)}$

Theorem

Let $f\colon \omega \to \omega$ be an increasing injection. The following conditions are equivalent:

(i) f is bi-$\mathcal{I}_d$-invariant;

(ii) $\underline{d}(f[\omega]) > 0$;

(iii) there is $C \in \omega$ such that $f(n) \leq Cn$ for every $n \geq 1$;

(iv) f is bi-$\mathcal{I}_{(1/n)}$-invariant.

Lower density

$$\underline{d}(A) = \liminf \frac{card(A \cap n)}{n}$$

Bibliography

M. Balcerzak, Sz. Głąb, J. Swaczyna, *Ideal invariant injections*, Preprint.

I. Farah, *Analytic quotients. Theory of lifting for quotients over analytic ideals on integers*, Mem. Amer. Math. Soc. **148** (2000).

D. Meza Alcantara, *Ideals and filters on countable sets*, PhD thesis, Univ. Nacional Autonoma de Mexico (June 2009).

Thank you for your attention!