Nonseparable growth of ω supporting a strictly positive measure

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Definitions

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Remark

There is a strictly positive measure on a Boolean algebra $\mathfrak{A}$ iff there is a strictly positive measure on the Stone space $\mathrm{ult}(\mathfrak{A})$.

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Lebesgue measure algebra

Let $\mathfrak{B} = Bor[0,1]/\mathscr{N}$, where $\mathscr{N} = \{A \subseteq [0,1] : \lambda(A) = 0\}$. It has nonseparable $\mathrm{ult}(\mathfrak{B})$ and the measure λ transfers to a strictly positive measure on $\mathrm{ult}(\mathfrak{B})$. Assuming CH, by Parovičenko $\mathrm{ult}(\mathfrak{B})$ embeds into $\mathscr{P}(\omega)/fin$, so it is a growth of ω.

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Dow & Hart: Under Open Coloring Axiom the measure algebra does not embed into $\mathscr{P}(\omega)/\mathit{fin}$.

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Is there a ZFC example of nonseparable growth of ω which supports a strictly positive measure?
Equivalently: is there a ZFC example of a Boolean algebra with nonseparable Stone space that supports a strictly positive measure and can be embedded into $\mathscr{P}(\omega)/\mathit{fin}$?

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- **Drygier & Plebanek:** under $\mathfrak{b} = \mathfrak{c}$ (or some weaker statement) an example of $\gamma\omega$ with nonseparable remaider supporting a strictly positive measure
- **Borodulin-Nadzieja & Inamdar:** ZFC example of nonseparable growth of ω supporting a strictly positive measure

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Measure algebra, continued

Frankiewicz & Gutek: Under CH, there is an embedding $\Phi : \mathfrak{B} \to \mathscr{P}(\omega)/fin$ such that $\lambda(b) = d(\Phi(b))$ for any $b \in \mathfrak{B}$.

Main result

Theorem

There exists a Boolean algebra $\mathfrak{A}$ with the following properties:

- $\mathrm{ult}(\mathfrak{A})$ is not separable
- there exists a strictly positive measure μ on $\mathfrak{A}$
- there exists an embedding $\Psi : \mathfrak{A} \to \mathscr{P}(\omega)/\mathit{fin}$ such that $\mu(a) = d(\Psi(a))$ for any $a \in \mathfrak{A}$.

Construction

Notations

$\{P_\alpha : \alpha < \mathfrak{c}\} = [2^\omega]^{\leq\omega}$
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Definition of algebra

$\mathfrak{A} = alg\left(\mathrm{Clop}(2^\omega) \cup \{U_\alpha : \alpha < \mathfrak{c}\}\right)$

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- we define for any $\alpha < \mathfrak{c}$ such $\Psi_0(U_\alpha)$ that $\lambda(U_\alpha) = d(\Psi_0(U_\alpha))$ and we can extend Ψ_0 to a homomorphism $\Psi : \mathfrak{A} \to \mathscr{P}(\omega)/fin$

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- $\Psi : \mathfrak{A} \to \mathscr{P}(\omega)/fin$ also transfers the Lebesgue measure to the asymptotic density
- the homomorphism Ψ is an embedding, which is an easy corollary from transferring the measure to density

Appendix: The other construction

Theorem (Borodulin-Nadzieja, Inamdar, 2015)

There is a Boolean algebra $\mathfrak{T} \subseteq \mathscr{P}(\omega)/\mathit{Fin}$ such that

- $\mathfrak{T}$ is not σ-centered,
- $\Vdash_{\mathfrak{B}}$ "$\check{\mathfrak{T}}$ is σ-centered".

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Theorem (Kamburelis, 80')

If $\mathfrak{C}$ is a Boolean algebra and $\Vdash_{\mathfrak{B}}$ "$\check{\mathfrak{C}}$ is σ-centered", then $\mathfrak{C}$ supports a strictly positive measure.