Compact sets in Euclidean spaces as IFS-attractors

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joint work with T. Banakh

Which sets can be embedd in Euclidean space?

Definition

A metric space (X, d) is called **doubling** if there exists a natural number M such that each open ball $B(x, r)$ is contained in the union of at most M open balls $B(y, \frac{r}{2})$.

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Assouad's theorem

For each doubling space (X, d) and for each $\alpha \in (0, 1)$ there exists $n \in \mathbb{N}$ and bi-Lipschitz function $\varphi\colon (X, d^{\alpha}) \to \mathbb{R}^n$.

Definition

An **Iterated Function System** (IFS) is a finite collection of contractions on the metric space X:

$$\mathcal{F} = \{f_1, f_2, \dots, f_n \colon X \to X;\ \max_{i=1,\dots,n} \{\mathrm{Lip} f_i\} < 1\}.$$

A nonempty compact set $A \subset X$ which is invariant by the IFS $\mathcal{F}$, in the sense:

$$A = f_1(A) \cup f_2(A) \cup \dots \cup f_n(A)$$

is called the **attractor of the IFS** $\mathcal{F}$ (IFS-attractor).

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Theorem

For every IFS on a complete metric space X there exist a unique IFS-attractor.

Being an IFS-attractor is not a topological invariant

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Problem

Which compact space is homeomorphic to an IFS-attractor **in the Euclidean space**?

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Definition

A compact space A is a **Euclidean fractal** if it is homeomorphic to some IFS-attractor in $\mathbb{R}^n$ (there exists a metric on A and IFS $\mathcal{F} = \{f : A \to A\}$ such that $A = \bigcup_{f \in \mathcal{F}} f(A)$).

Being an IFS-attractor is a bi-Lipschitz invariant

Fact

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$$\begin{array}{ccc} X & \xrightarrow{\mathcal{F}} & X \\ \uparrow{\scriptstyle \varphi^{-1}} & & \downarrow{\scriptstyle \varphi} \\ \varphi(X) & \xrightarrow{\mathcal{F}'} & \varphi(X) \end{array}$$

X - IFS-attractor for family $\mathcal{F}$ and each $f \in \mathcal{F}$ is λ-Lipschitz in X $(\lambda < 1)$.

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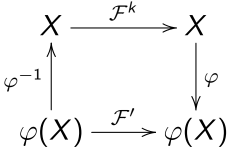

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For every $k \in \mathbb{N}$, X is an IFS-attractor for the family $\mathcal{F}^k = \{f_1 \circ \cdots \circ f_k \colon f_1, \ldots, f_k \in \mathcal{F}\}$ of a λ^k-Lipschitz function.

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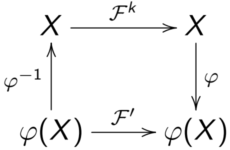

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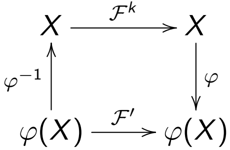

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Take a $k \in \mathbb{N}$ such that $\mathrm{Lip}\ \varphi \cdot \lambda^k \cdot \mathrm{Lip}\ \varphi^{-1} < 1$ then $\varphi(X)$ is an IFS-attractor for $\mathcal{F}' = \{\varphi \circ f_1 \circ \cdots \circ f_k \circ \varphi^{-1} \colon f_1, \ldots, f_k \in \mathcal{F}\}$

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(X,d) is an IFS-attractor for λ-Lipschitz function from $\mathcal{F}$ $(\lambda < 1)$.

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$\varphi(X) \subset \mathbb{R}^n$ is an IFS-attractor.

Theorem (Banakh, N 2015)

Let X be compact doubling space and Z be its uncountable, zero-dimensional, subset open in X. Then X is an Euclidean fractal.

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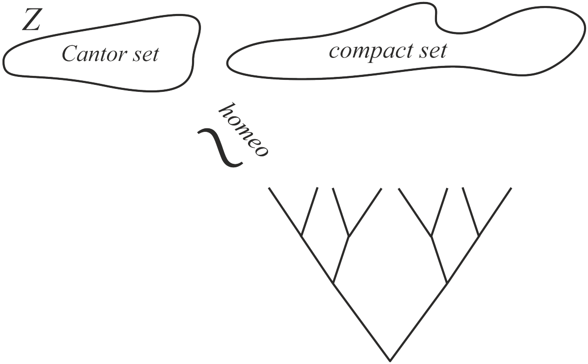

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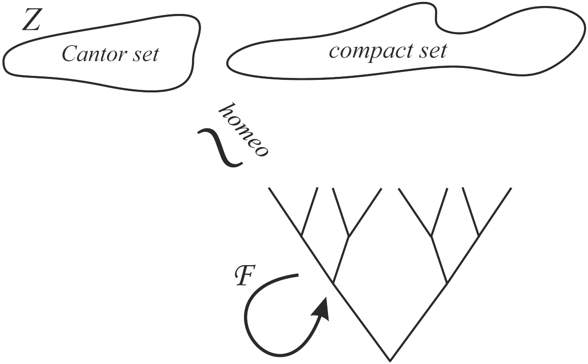

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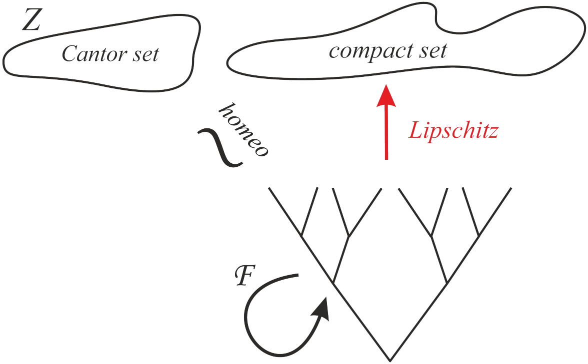

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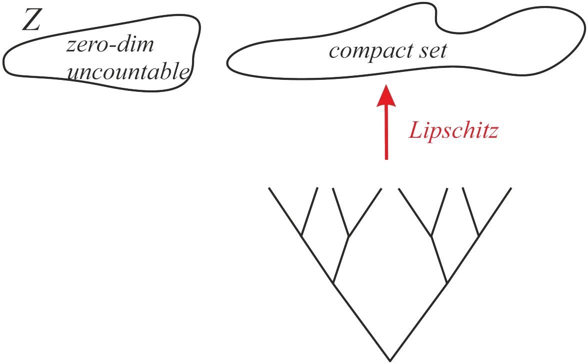

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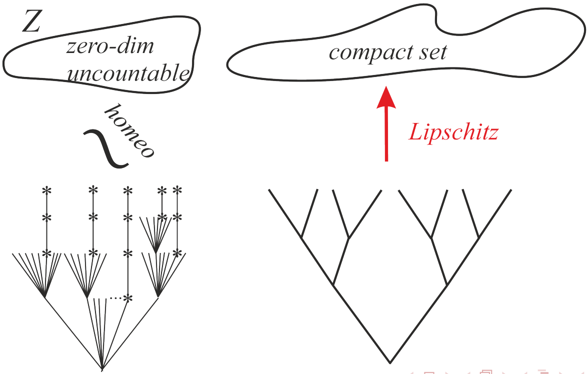

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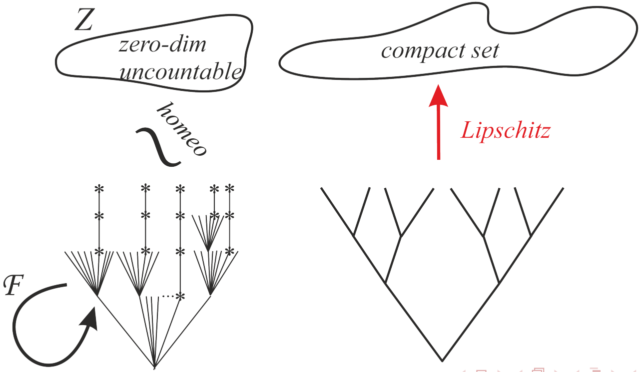

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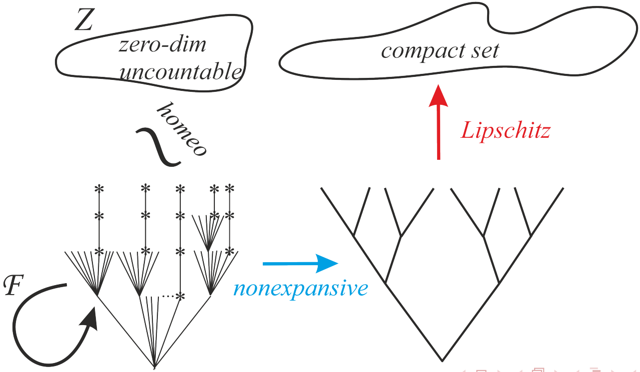

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