On maximal connected topologies

Adam Bartoš
drekin@gmail.com

Faculty of Mathematics and Physics
Charles University in Prague

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Maximal and minimal topologies

Definition

Let X be a set. The set of all topologies on X is a complete lattice denoted by $\mathcal{T}(X)$.
Let $\mathcal{P}$ be a property of topological spaces.

- We say a topology $\tau \in \mathcal{T}(X)$ is *maximal* $\mathcal{P}$ if it is a maximal element of $\{\sigma \in \mathcal{T}(X) : \sigma \text{ satisfies } \mathcal{P}\}$, i.e. τ satisfies $\mathcal{P}$ but no strictly finer topology satisfies $\mathcal{P}$. In that case $\langle X, \tau \rangle$ is a *maximal* $\mathcal{P}$ *space*.
- We say a topology $\tau \in \mathcal{T}(X)$ is *minimal* $\mathcal{P}$ if it satisfies $\mathcal{P}$ but no strictly coarser topology satisfies $\mathcal{P}$. In that case $\langle X, \tau \rangle$ is a *minimal* $\mathcal{P}$ *space*.

Maximal and minimal topologies

Examples

- *Maximal space* means *maximal without isolated points*.
- A compact Hausdorff space is both *maximal compact* and *minimal Hausdorff*.
- We are interested in *maximal connected spaces*.

For more examples see [Cameron, 1971].

Maximal connected topologies were first considered by Thomas in [Thomas, 1968]. Thomas proved that an open connected subspace of a maximal connected space is maximal connected, and characterized principal maximal connected spaces.

Submaximal spaces

Definition

Recall that a topological space is *submaximal* if every its dense subset is open. Equivalently, if every its subset is an intersection of an open set and a closed set.

A topological space is $T_{\frac{1}{2}}$ if every its singleton is open or closed.

Proposition

We have the following implications.

- submaximal $\implies$ $T_{\frac{1}{2}}$
- maximal connected $\implies$ submaximal
- T_1 maximal $\implies$ submaximal

Subspaces of maximal connected spaces

Lemma

Let $\langle Y, \sigma \rangle$ be a subspace of a connected space $\langle X, \tau \rangle$. For every connected expansion $\sigma^* \geq \sigma$ there exists a connected expansion $\tau^* \geq \tau$ such that $\tau^* \restriction Y = \sigma^*$.

Sketch of the proof.
We put $\tau^* := \tau \vee \{S \cup (X \setminus \overline{Y}) : S \subseteq Y \ \sigma^*\text{-open}\}$.

Corollary [Guthrie–Reynolds–Stone, 1973]

Maximal connectedness is preserved by connected subspaces.

Principal spaces and specialization preorder

Definition

Let X be a topological space.

- X is called *principal* or *finitely generated* if every intersection of open sets is open. Equivalently, if $\overline{A} = \bigcup_{x \in A} \overline{\{x\}}$, $A \subseteq X$.
- The *specialization preorder* on X is defined by

$$x \leq y \;:\Longleftrightarrow\; \overline{\{x\}} \subseteq \overline{\{y\}}.$$

Facts

- Every open set is an upper set. Every closed set is a lower set.
- The converse holds if and only if the space is principal.
- The specialization preorder is an order if and only if the space is T_0.
- Every isolated point is a maximal element, every closed point is a minimal element.

Principal maximal connected topologies

Let X be a principal $T_{\frac{1}{2}}$ space.

- The topology is uniquely determined by the specialization preorder, which is an order with at most two levels.
- Let us consider a graph G_X on X such that there is an edge between $x, y \in X$ if and only if $x < y$ or $y < x$.
- X is connected $\iff$ G_X is connected as a graph.
- X is maximal connected $\iff$ G_X is a tree.

Therefore, principal maximal connected spaces correspond to trees with fixed bipartition.

Examples

The empty space, the one-point space, the Sierpiński space, principal ultrafilter spaces, principal ultraideal spaces.

Maximal connected topologies in other special classes

Maximal connected spaces can be characterized also in the following special classes of topological spaces:

- door spaces,
- hyperconnected spaces,
- ultraconnected spaces,
- extremally disconnected spaces,
- maximal spaces.

These include only free ultrafilter spaces, principal ultrafilter spaces, principal ultraideal spaces, none of which is Hausdorff.

Strongly connected and essentially connected topologies

Definition

A topological spaces is called

- *strongly connected* [Cameron, 1971] if it has a maximal connected expansion;
- *essentially connected* [Guthrie–Stone, 1973] if it is connected and every connected expansion has the same connected subsets.

Observation

- Every maximal connected space is both strongly connected and essentially connected.
- Every connected subspace of an essentially connected space is such.
- Every connected subspace of an essentially connected strongly connected space is such.

Hausdorff non-strongly connected topologies

Is there an infinite Hausdorff connected space that has no maximal connected expansion?

Definition

Let X be a topological space. Recall that a point $x \in X$ is

- a *cutpoint* if $X \setminus \{x\}$ is disconnected,
- a *dispersion point* if $X \setminus \{x\}$ is hereditarily disconnected.

Theorem [Guthrie–Stone, 1973]

No Hausdorff connected space with a dispersion point has a maximal connected expansion.

- A dispersion point is the only cutpoint of a connected space.
- Every infinite Hausdorff maximal connected space has infinitely many cutpoints.
  - Every nonempty nondense open subset of a maximal connected space contains a cutpoint in its boundary.

Hausdorff non-strongly connected topologies

Examples

Hausdorff connected spaces with a dispersion point include

- Roy's countable space,
- Knaster–Kuratowski fan / Cantor's leaky tent.

Observation

Not every connected subspace of a strongly connected space is such since Knaster–Kuratowski fan is a subspace of $\mathbb{R}^2$.

Hausdorff maximal connected topologies

Is there an infinite Hausdorff maximal connected space?
[Thomas, 1968]

Theorem [Simon, 1978] and [Guthrie–Stone–Wage, 1978]

There exists a maximal connected expansion of the real line.

Corollary

There exists an functionally Hausdorff maximal connected space of size $\mathfrak{c}$.

Theorem [El'kin, 1979]

For every $\kappa \geq \omega$ there exists a Hausdorff maximal connected space X such that $\Delta(X) = |X| = \kappa$.

Strongly connected and essentially connected topologies

Theorem [Hildebrand, 1967]

The real line is essentially connected.

Corollary

The spaces $\mathbb{R}$, $[0,1)$, $[0,1]$ are both strongly connected and essentially connected.

Tree sums of topological spaces

Definition

Let $\langle X_i : i \in I \rangle$ be an indexed family of topological spaces, $\sim$ an equivalence on $\sum_{i \in I} X_i$, and $X := \sum_{i \in I} X_i / \sim$. We consider

- the canonical maps $e_i \colon X_i \to X$,
- the canonical quotient map $q \colon \sum_{i \in I} X_i \to X$,
- the *set of gluing points* $S_X := \{x \in X : |q^{-1}(x)| > 1\}$,
- the *gluing graph* G_X with vertices $I \sqcup S_X$ and edges of from $s \to_x i$ where $s \in S_X$, $i \in I$, and $x \in X_i$ such that $e_i(x) = s$.

We say that X is a *tree sum* if G_X is a tree, i.e. for every pair of distinct vertices there is a unique undirected path connecting them.

Example

A *wedge sum*, that is a space $\sum_{i \in I} X_i / \sim$ such that one point is chosen in each space X_i and $\sim$ is gluing these points together, is an example of a tree sum.

Tree sums of topological spaces

Proposition

A topological space X is naturally homeomorphic to a tree sum of a family of its subspaces $\langle X_i : i \in I \rangle$ if and only if the following conditions hold.

1. $\bigcup_{i \in I} X_i = X$,
2. X is inductively generated by embeddings $\{e_i \colon X_i \to X\}_{i \in I}$,
3. G is a tree, where G is a graph on $S \sqcup I$ satisfying
   - $S := \{x \in X : |\{i \in I : x \in X_i\}| \geq 2\}$,
   - $s \to i$ is an edge if and only if $s \in S$, $i \in I$, and $s \in X_i$.

Proposition

Let X be a tree sum of spaces $\langle X_i : i \in I \rangle$. X is separated if and only if all the spaces X_i are separated for "separated" meaning T_0, symmetric, T_1, T_2, $T_{2\frac{1}{2}}$, functionally T_2, totally separated, regular, completely regular, or zero-dimensional.

Tree sums of topological spaces

Proposition

Let X be a tree sum of spaces $\langle X_i : i \in I \rangle$.

1. X is connected if and only if all the spaces X_i are connected.

Let $C \subseteq X$ and suppose that every gluing point of X is closed.

2. C is connected if and only if every C_i is connected and G_C is connected (i.e. it is a subtree of G_X), where
   - $C_i := C \cap X_i$ for $i \in I$,
   - $I_C := \{i \in I : C_i \neq \emptyset\}$,
   - $S_C := S_X \cap C$,
   - G_C is the subgraph of G_X induced by $I_C \sqcup S_C$.

   In this case, C is the induced tree sum of spaces $\langle C_i : i \in i \rangle$.

Tree sums of maximal connected spaces

Proposition

Let $\langle X, \tau \rangle := \sum_{i \in I} \langle X_i, \tau_i \rangle / {\sim}$ be a tree sum, $\mathcal{A} \subseteq \mathcal{P}(X)$. We put $\tau^* := \tau \vee \mathcal{A}$, $\tau_i^* := \tau_i \vee \{A \cap X_i : A \in \mathcal{A}\}$ for $i \in I$. If

- the set of gluing points S_X is closed discrete in $\langle X, \tau \rangle$,
- the family $\mathcal{A}$ is point-finite at every point of S_X,

then $\langle X, \tau^* \rangle = \sum_{i \in I} \langle X_i, \tau_i^* \rangle / {\sim}$, i.e. such expansion of a tree sum is a tree sum of the corresponding expansions.

Theorem

Let X be a tree sum of spaces $\langle X_i : i \in I \rangle$ such that the set of gluing points is closed discrete.

1. If the spaces X_i are maximal connected, then X is such.
2. If the spaces X_i are strongly connected, then X is such.
3. If the spaces X_i are essentially connected, then X is such.

Tree sums of maximal connected spaces

Examples

As a corollary we have that the spaces like $\mathbb{R}^\kappa$, $[0,1]^\kappa$, $\mathbb{S}^n$ are are strongly connected, and every topological tree graph is both strongly connected and essentially connected.

Observation

If $\langle X, \tau \rangle$ is a topological tree graph, $\langle X, \tau^* \rangle$ its maximal connected expansion, $x \in X$, then the number of components of $X \setminus \{x\}$ is the same with respect to τ as with respect to τ^*.

Example

Every maximal connected principal space is a tree sum of copies of the Sierpiński space. Note that in this case the set gluing points does not have to be discrete or closed.

Tree sums of maximal connected spaces

Example

Consider a comb-like space $\langle X, \tau \rangle := \sum_{x \in [0,1]} [0,1]_x / \sim$ where $[0,1]_x$ are copies of the real interval $[0,1]$ with a maximal connected expansion of the standard topology, and $\sim$ glues together points $\langle 0, x \rangle \sim \langle x, 1 \rangle$ for $x > 0$.

$\langle X, \tau \rangle$ is a tree sum of the maximal connected intervals, but it is not maximal connected itself.

Let $A := \langle 0, 0 \rangle \cup \bigcup_{x>0} [0,1)_x$, $\tau^* := \tau \vee \{A\}$. A is not τ-open, but τ^* is still connected. Since $[0,1]_0$ becomes disconnected, τ is not even essentially connected.

Observation

A tree sum of maximal connected spaces is maximal connected if and only if it is essentially connected.

Regular maximal connected topologies

Open problems

- Is there an infinite regular maximal connected space?
- Is there an infinite regular submaximal connected space?
- Is there an infinite regular irresolvable connected space?

For more related open problems see [Pavlov, 2007].

Definition

Recall that a topological space is called *resolvable* if there exist two disjoint dense subsets. Otherwise, it is called *irresolvable*.

Fact

We have

$$\text{submaximal} \implies \text{hereditarily irresolvable} \implies \text{irresolvable.}$$

Regular maximal connected topologies

Theorem [Alas–Sanchis–Tkačenko–Tkachuk–Wilson, 2000]

The following conditions are equivalent in ZFC.

1. There exists an irresolvable Baire space without isolated points.
2. There exists a submaximal space that is not σ-discrete.
3. There exists a maximal space that is not σ-discrete.

If $V = L$, then every Baire space without isolated points is resolvable, and hence every submaximal space is σ-discrete. Moreover, if the space is normal, then it is zero-dimensional as a countable union of closed zero-dimensional subsets.

Corollary

($V = L$) There is no infinite normal maximal connected space.

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Thank you for your attention.