Rosenthal families and the Grothendieck property

Damian Sobota

Institute of Mathematics, Polish Academy of Sciences

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Of course!

If $\omega = \bigcup_{k\in\omega} N_k$ is a partition ($N_k \in [\omega]^\omega$), then:

$$\sum_k \mu\Big(\bigcup_{n\in N_k} a_n\Big) \leqslant \mu(\omega) < \infty$$

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No! Unfortunately...

If $\mu_k = \delta_k$ and $a_n = \{n\}$ and $\varepsilon = 1/2$, then:

$$\mu_{n_k}\Big(\bigcup_{l\in\omega} a_{n_l}\Big) = \mu_{n_k}\Big(\bigcup_{l\neq k} a_{n_l}\Big) + \mu_{n_k}(a_{n_k}) = 0 + 1 > 1/2 = \varepsilon$$

Rosenthal's lemma

Theorem (Rosenthal '70)

Let $(a_n : n \in \omega)$ be an antichain in $\wp(\omega)$. Assume (μ_k) is a sequence of positive finitely additive measures on $\wp(\omega)$ satisfying the inequality $\mu_k\left(\bigcup_{n\in\omega} a_n\right) < 1$ for every $k \in \omega$. Fix $\varepsilon > 0$.

Then, there exists an infinite set $A \subseteq \omega$ such that for every $k \in A$ the following inequality is satisfied:

$$\mu_k\Big(\bigcup_{\substack{n\in A\\ n\neq k}} a_n\Big) < \varepsilon.$$

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Then, there exists an infinite set $A \subseteq \omega$ *such that for every* $k \in A$ *the following inequality is satisfied:*

$$\mu_k\Big(\bigcup_{\substack{n\in A\\ n\neq k}} a_n\Big) < \varepsilon.$$

Question: Can we control the choice of A?

Rosenthal families

Definition

Let $\mathcal{F} \subseteq [\omega]^\omega$. $\mathcal{F}$ is called ***Rosenthal*** if for every antichain (a_n) on ω, sequence (μ_k) of positive measures on ω such that $\mu_k\left(\bigcup_{n\in\omega} a_n\right) < 1$ for every $k \in \omega$, and $\varepsilon > 0$, there is $A \in \mathcal{F}$ such that:

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Fact

1. $\omega_1 \leqslant \mathfrak{ros} \leqslant \mathfrak{c}$.

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Fact

1. $\omega_1 \leqslant \mathfrak{ros} \leqslant \mathfrak{c}$.
2. *Assuming MA,* $\mathfrak{ros} = \mathfrak{c}$.

Selective ultrafilters

Definition

Let $\mathcal{F} \subseteq [\omega]^\omega$ be a non-principal ultrafilter. $\mathcal{F}$ is ***selective*** (also ***Ramsey***) if for every partition $\omega = \bigcup_{k\in\omega} N_k$ ($N_k \in \wp(\omega) \setminus \mathcal{F}$) there is $F \in \mathcal{F}$ such that $|F \cap N_k| = 1$ for every $k \in \omega$.

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Theorem (Rudin '56)

Assuming CH, there is a selective ultrafilter.

Theorem (Shelah '82)

There is a model of ZFC without selective ultrafilters.

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Theorem (S.)

Assume $\mathcal{U}$ is a base of a selective ultrafilter. Then $\mathcal{U}$ is Rosenthal.

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So $\mathfrak{u}_s \geqslant \mathfrak{ros}$.

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There is a model of ZFC in which $\mathfrak{u}_s = \omega_1 < \omega_2 = \mathfrak{c}$.

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Corollary

It is consistent that $\omega_1 = \mathfrak{ros} < \mathfrak{c}$.

An immediate application: operators from ℓ_∞

$\ell_\infty = \{x \in \mathbb{R}^\omega : \ \|x\|_\infty := \sup_{n\in\omega} |x(n)| < \infty\}$

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Let X be a Banach space, $T : \ell_\infty \to X$ a continuous operator such that $T|_{c_0}$ is an isomorphism.

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   Then there is such $M \in \mathcal{F}$ that $T|_{\ell_\infty(M)}$ is an isomorphism.
2. *Let $\mathcal{F} \subseteq [\omega]^\omega$ be* ***a base of a selective ultrafilter****.*
   Let X be a Banach space...

Weak topologies on Banach spaces

X – a Banach space

X^* – ***the dual*** of X – the space of continuous functionals on X

X^{**} – ***the bidual*** of X – the space of continuous functionals on X^*

$X \hookrightarrow X^{**}$ by $x \mapsto ev_x$ where $ev_x(x^*) = x^*(x)$ for $x^* \in X^*$

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The Grothendieck property

Definition

A Banach space X has ***the Grothendieck property*** if every weak* convergent sequence $(x_n^* \in X^* : n \in \omega)$ is weakly convergent.

Notable examples

1. reflexive spaces, e.g. ℓ_p for $1 < p < \infty$

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4. $C(St(\mathcal{A}))$ if $|\mathcal{A}| \leqslant \max(\mathfrak{s}, \mathrm{cov}(\mathcal{M}))$

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Theorem (Brech '06)

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Small algebras with the Grothendieck property

Definition (The κ-anti-Grothendieck property)

Let κ be a cardinal number. A Boolean algebra $\mathcal{A}$ has ***the κ-anti-Grothendieck property*** if there exists a family $\{(a_n^\gamma \in \mathcal{A} : n \in \omega) : \gamma < \kappa\}$ of κ many antichains in $\mathcal{A}$ with the following property:

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Definition

$\mathfrak{g}_a = \min\{\kappa : \text{every ctbl } \mathcal{A} \text{ has the } \kappa\text{-anti-Grothendieck property}\}$.

Fact

$\mathfrak{b} \leqslant \mathfrak{g}_a \leqslant \mathfrak{c}$.

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Theorem (S.)

If κ *is a cardinal such that* $\kappa \geqslant \max(\mathfrak{g}_a, \mathfrak{ros})$ *and* $\mathrm{cof}([\kappa]^\omega) = \kappa$*,*

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Problem

Estimate (determine!) the value of $\mathfrak{g}_a$.

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Conjecture: $\mathfrak{g}_a \leqslant \mathrm{cof}(\mathcal{N})$.

Almost the end...

Thank you for your attention...

Transfinite methods in Banach spaces and algebras of operators

Confirmed speakers:

Dales, Dow, Godefroy, Todorčević...

https://www.impan.pl/~set_theory/Banach2016/