Winter School in Abstract Analysis section Topology

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# On skeletal maps

by

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Used in Katowice, a short form of the definition of a skeletal map is:

(I) Skeletal means that the images of regularly closed subsets are regularly closed.

See J. Mioduszewski MR0810825 (87h:54070)]. J. Mioduszewski and L. Rudolf observed [see Dissertationes Math. Rozprawy Mat. 66 (1969): *H-closed and extremally disconnected Hausdorff spaces*] that skeletal maps are suitable for the adjoint functor in the theory of Katetov's $H$-closed extensions of (Hausdorff) topological spaces, (1940) - see MR0001912 (1,317i) or (1947) - see MR0022069 (9,153d). For me, a user-friendly definition is:

(II) A continuous function $f : X \to Y$ is called *skeletal*, whenever $\mathrm{Int}_Y \, \mathrm{cl}_Y \, f[V] \neq \emptyset$ for any open $V \subseteq X$.

## **Two exercises**:

(I)$\Longrightarrow$ (II), whenever $f : X \to Y$ is continuous.

Indeed, suppose $V \subseteq X$ is a non-empty open set which witnesses $\neg(II)$. Thus $\mathrm{Int}_Y \, \mathrm{cl}_Y \, f[V] = \emptyset$. Also $f[\mathrm{cl}_X \, V] \subseteq \mathrm{cl}_Y \, f[V]$, since $f$ is continuous. Hence $\mathrm{Int}_Y \, f[\mathrm{cl}_X \, V] = \emptyset$. So, the image of reguraly closed set $\mathrm{cl}_X \, V$ is not reguraly closed. $\square$

If $X$ is a quasi-regular topological space, then (II) $\Longrightarrow$ (I), whenever $f : X \to Y$ is a continuous and closed function.

Indeed, suppose $V \subseteq X$ is a non-empty open set which witnesses $\neg(I)$. Thus $f[\mathrm{cl}_X \, V]$ is a closed set and $f[\mathrm{cl}_X \, V] \setminus \mathrm{cl}_Y \, \mathrm{Int}_Y \, f[\mathrm{cl}_X \, V] = W \neq \emptyset$. Take an open set $U = V \cap f^{-1}(Y \setminus \mathrm{cl}_Y \, \mathrm{Int}_Y \, f[\mathrm{cl}_X \, V])$. Thus $f[U] \subseteq W$ and $f[U] \subseteq \mathrm{Int}_Y \, f[\mathrm{cl}_X \, V]$. Hence $\mathrm{Int}_Y \, f[U] = \emptyset$. But $X$ is quasi-regular, so any non-empty open set with the closure contained in $U$ witnesseses $\neg(II)$. $\square$

Thus, for compact Hausdorff topological spaces one can use the following definition:

**Semi-open, when compact and Hausdorff is not assumed**

Suppose $f : X \to Y$ is a continuous function, where $X$ and $Y$ are compact and Hausdorff. If $\mathrm{Int}_Y f[V] \neq \emptyset$ for any open $V \subseteq X$, then $f$ is called skeletal.

There are a few other possibilities to introduce skeletal maps: Not equivalent in general, but equivalent under some restrictions. For examples:

**Original definition by M. Henriksen and M. Jerison (1965)**

Whenever $\mathrm{Int}_X \mathrm{cl}_X f^{-1}(U) = \mathrm{Int}_X f^{-1}(\mathrm{cl}_y U)$, for any open $U \subseteq Y$.

**Almost-open by A. Arhangelskij (1961).**

Whenever $f : X \to Y$ is a continuous function such that each non-empty open subset $U \subseteq X$ contains a non-empty open subset $V \subseteq U$ with open image $f[V] \subseteq Y$.

**A. Błaszczyk, Coll. Math. 32 (1974)**

Whenever $f : X \to Y$ is a continuous function such that perimage (under $f$) of any open and dense subset of $Y$ is dense in $X$.

## Complete embeddings - regular subalgebras

Let $\mathcal{P}$ be an ordered (partially) set and $\mathcal{Q} \subseteq \mathcal{P}$ be such that any incompabile elements in $\mathcal{Q}$ are incompabile in $\mathcal{P}$. Then $\mathcal{Q}$ is *complete embeding* in $\mathcal{P}$, whenever $W \subseteq \mathcal{Q}$ is predense in $\mathcal{Q}$ if, and only if $W$ is predense in $\mathcal{P}$.
When $\mathcal{Q}$ is *complete embeding* in $\mathcal{P}$, then we write $\mathcal{Q} \subseteq_c \mathcal{P}$.

When $\mathcal{Q}$ and $\mathcal{P}$ are Boolean Algebras, then $\mathcal{Q} \subseteq_c \mathcal{P}$ means that $\mathcal{Q}$ is a regular subalgebra of $\mathcal{P}$.

I do not know: Who first considered notions of regular subalgebra! Symbol $\mathcal{Q} \subseteq_c \mathcal{P}$ complete embedings were used in P. Daniels, K. Kunen and H. Zhou, Fund. Math. 145 (1994).

## Inclusion

When the inclusion is a considered order, then

compatible means non-empty intersection,

incompatible means empty intersection,

$\mathcal{Q} \subseteq_c \mathcal{P}$ means: For any $W \in \mathcal{P}$ there exists $V \in \mathcal{Q}$ such that $U \subseteq V$ and $\emptyset \neq U \in \mathcal{Q}$ implies $U \cap W \neq \emptyset$.

**Proposition**

*Let $\mathcal{T}_X$ be the family of all non-empty open subsets of $X$ and $\mathcal{B}$ be a $\pi$-base for $Y$. A continuous function $f : X \to Y$ is skeletal if, and only if $\{f^{-1}(V) : V \in \mathcal{B}\} \subseteq_c \mathcal{T}_X$.*

*Proof*. Consider a skeletal map $f$ and $V \in \mathcal{T}_X$. Take $W \in \mathcal{B}$ such that $\emptyset \neq W \subseteq \mathrm{Int}_Y \mathrm{cl}_Y f[V]t$. If $U \in \mathcal{B}$ and $\emptyset \neq U \subseteq W$, then $U \subseteq \mathrm{cl}_Y f[V]$. Hence $U \cap f[V] \neq \emptyset$ and $f^{-1}(U) \cap V \neq \emptyset$.

Suppose $U \in \mathcal{T}_X$ and $\mathrm{Int}_Y \, \mathrm{cl}_Y \, f[U] = \emptyset$. Since $\mathcal{B}$ is a $\pi$-base, for each non-empty $W \in \mathcal{B}$ there exists a non-empty set $V \in \mathcal{B}$ such that $V \subseteq W$ and $V \cap f[U] = \emptyset$. Thus $f^{-1}(V) \cap U = \emptyset$, i.e. $U$ witnesses that $\{f^{-1}(V) : V \in \mathcal{B}\}$ is not colmplete embeding of $\mathcal{T}_X$. $\square$

Suppose $\mathcal{B}$ is a family of sets. We say that a family $\mathcal{C} \subseteq [\mathcal{B}]^{\leqslant\omega}$ is a *club* in $\mathcal{B}$, whenever $\mathcal{C}$ is closed under increasing sequence and for each $A \in [\mathcal{B}]^{\leqslant\omega}$ there exists $D \in \mathcal{C}$ such that $A \subseteq D$.

**P. Daniels, K. Kunen and H. Zhou, Fund. Math. 145 (1994), Theorem 1.6**

We say that a compact Hausdorff space $X$ is I-*favorable*, whenever the family $\{\mathcal{P} \in [\mathcal{T}_X]^{\leqslant\omega} : \mathcal{P} \subseteq_c \mathcal{T}_X\}$ contains a club in $\mathcal{T}_X$.

A join with Andrzej Kucharsk and published result is:

**Top. Appl. (2008) A. Kucharski and Sz. Plewik**

A compact Hausdorff space $X$ is a I-favorable if, and only if there exists a $\sigma$-complete inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ - where all spaces $X_\sigma$ are compact and metrizable, and all bonding maps $\pi^\sigma_\varrho$ are skeletal - such that $X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$.

Now we can add:

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If $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ a $\sigma$-complete inverse sequence of I-favorable compact Hausdorff spaces - where all bonding maps $\pi^\sigma_\varrho$ are skeletal and onto, then the inverse limit space $X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is I-favorable (also compact and Hausdorff).

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Any a compact non-metrizable and $I$-favorable space $X$ is homeomorphic with the inverse limit of a continuous sequence $\{X_\alpha; p_\alpha^\beta; \omega \leqslant \alpha < \beta < \mathrm{w}(X)\}$, where each $X_\alpha$ is a compact Hausdorff and $I$-favorable space, with $\mathrm{w}(X_\alpha) < \mathrm{w}(X)$, and such that any bounding map $p_\alpha^\beta$ is skeletal.

Two last results are proved similar to the first, i.e. using terminology of the open-open game and applying Frink's characterization of completly regular spaces. One can find Boolean Algebra counterparts of the above results in B. Balcar, T. Jech and J. Zapletal, *Semi-Cohen Boolean algebras*, Ann. Pure Appl. Logic 87 (1997) or L. Heindorf and L. Shapiro, *Nearly projective Boolean algebras*, Lecture Notes in Mathematics (1994).

## Very I-favorable and d-open maps

Suppose $\mathcal{Q} \subseteq \mathcal{P}$ are families of open subsets of $X$. Again following P. Daniels, K. Kunen and H. Zhou, Fund. Math. 145, we write $\mathcal{Q} \subseteq_! \mathcal{P}$, whenever for any $\mathcal{S} \subseteq \mathcal{Q}$ and $x \notin \mathrm{cl}_X \cup \mathcal{S}$, there exists $W \in \mathcal{Q}$ such that $x \in W$ and $\emptyset = W \cap \mathrm{cl}_X(\cup \mathcal{S})$.

$\mathcal{Q} \subseteq_! \mathcal{P}$ implies $\mathcal{Q} \subseteq_c \mathcal{P}$.

Indeed, if $W \in \mathcal{P}$, then let $S = \{V \in \mathcal{Q} : V \cap W = \emptyset\}$. Thus $\emptyset = W \cap \mathrm{cl}_X \cup \mathcal{S}$. Hence no $W \in \mathcal{P}$ witnesseses that $\mathcal{Q}$ is not clomplete embeding of $\mathcal{P}$.

The class of d-open maps collaborating with the relation " $\subseteq_!$ " similarly as skeletal maps with " $\subseteq_c$ ".

The notion of d-open maps was introduced by M. G. Tkachenko. In the end of MR0647029 (83d:54015b) W. Kulpa wrote: The notion of $d$-open map $f$ is introduced by the author and means that $f(A)$ is a dense subset of an open set for any open set $A$.

## Suitable definition of d-open maps

A continuous function $f : X \to Y$ is called d-*open*, whenever $\mathrm{cl}_X\, f^{-1}[V] = f^{-1}(\mathrm{cl}_Y\, V)$ for any open $V \subseteq Y$.

## Proposition

A continuous function $f : X \to Y$ is d-open if, and only if $\{f^{-1}(V) : V \in \mathcal{T}_Y\} \subseteq_! \mathcal{T}_X$.

## Proposition

A closed and continuous function $f : X \to Y$ is d-open if, and only if $f$ is an open map.

## P. Daniels, K. Kunen and H. Zhou, Fund. Math. 145 (1994)

We say that a compact Hausdorff space $X$ is *very* I-*favorable*, whenever the family $\{\mathcal{P} \in [\mathcal{T}_X]^{\leqslant\omega} : \mathcal{P} \subseteq_! \mathcal{T}_X\}$ contains a club in $\mathcal{T}_X$.

When we can change " $\subseteq_C$ " onto" $\subseteq_!$ " and $\mathcal{T}_X$ onto the family of of all open and cozero sets in $X$, then we obtain a charcterization of compact openly generated spaces:

**Theorem (with A. Kucharski)**

A compact Hausdorff space $X$ is openly genetrated, whenever the family $\{\mathcal{P} \in [\mathcal{T}]^{\leqslant\omega} : \mathcal{P} \subseteq_! \mathcal{T}\}$ contains a club in $\mathcal{T}$, where $\mathcal{T}$ is the family of all open and cozero sets in $X$.

Openly genetrated spaces are introduced by E.V. Shchepin (1976). In fact, if there exists a $\sigma$-complete inverse system $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ such that all spaces $X_\sigma$ are compact and metrizable, and all bonding maps $\pi^\sigma_\varrho$ are open and onto, then $X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is openly genetrated.

Also, we have a characterizatin of very I-favorable compact spaces:

**Theorem (with A. Kucharski)**

A compact space $X$ is very I-favorable if, and only if $X = \varprojlim\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$, where $\{X_\sigma, \pi^\sigma_\varrho, \Sigma\}$ is a $\sigma$-complete inverse system such that all spaces $X_\sigma$ are compact and have countable weight, but all bonding maps $\pi^\sigma_\varrho$ are d-open and onto.

Methods, for prooving last results, are (very) similar to that with the open-open game and skeletal maps!

## A folklore:

J. Mioduszewski said to me that previously, he tried to use the name Henriksen-Jerison map. This gives the abbreviation a *HJ*-map. But, such an abbreviation has well known nazi connotes and it had to be changed. Despite of such connotations, some author's used the abbreviation *HJ*-maps. For example, Coll. Math. 32 (1975), page 187.