Wadge Hierarchy on Second Countable Spaces

Reduction via relatively continuous relations

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Classify Definable subsets of topological spaces

X a 2nd countable T_0 topological space:

- A countable basis of open sets,
- Two points which have same neighbourhoods are equal.

Borel sets are naturally classified according to their definition

$$\mathbf{\Sigma}^0_1(X) = \{O \subseteq X \mid X \text{ is open}\},$$

$$\mathbf{\Sigma}^0_2(X) = \left\{\bigcup_{i\in\omega} B_i \;\middle|\; B_i \text{ is a Boolean combination of open sets}\right\},$$

$$\mathbf{\Pi}^0_\alpha(X) = \{A^\complement \mid A \in \mathbf{\Sigma}^0_\alpha(X)\},$$

$$\mathbf{\Sigma}^0_\alpha(X) = \left\{\bigcap_{i\in\omega} P_i \;\middle|\; P_i \in \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X)\right\}, \quad \text{for } \alpha > 2.$$

$$\text{Borel subsets of } X = \bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha(X)$$

Wadge reducibility

Let X be a topological space, $A, B \subseteq X$.
A is Wadge reducible to B, or $A \leq_W B$,
if there is a continuous function $f : X \to X$
that reduces A to B, i.e. such that
$f^{-1}(B) = A$ or equivalently

$$\forall x \in X \quad (x \in A \longleftrightarrow f(x) \in B).$$

Bill Wadge

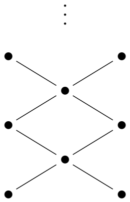
The idea is that the continuous function f reduces
the *membership question* for A to the membership question for B.

- The identity on X is continous, and
- continuous functions compose, so

Wadge reducibility is a quasi order on subsets of X. Is it useful?

Hierarchies?

On Polish 0-dimensional spaces,
the relation $\leq_W$ yields
a nice and useful hierarchy,
by results of Wadge, Martin, Monk,
Louveau, Duparc and others.



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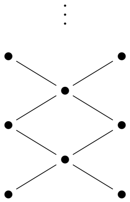
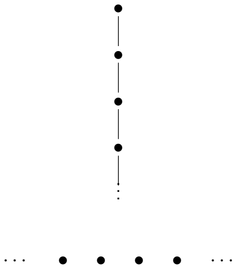


Thanks to a game theoretic formulation of the reduction.

On non 0-dim metric spaces, and many other non metrisable spaces the relation $\leq_W$ yields no hierarchy at all, by results of Schlicht, Hertling, Ikegami, Tanaka, Grigorieff, Selivanov and others.



The nice picture is lost...

Reduction by continuous functions yield a nice hierarchy of subsets of Polish 0-dimensional spaces.
To get a nice hierachy outside the realm of Polish 0-dim spaces:

- Motto Ros, Schlicht and Selivanov have considered reducibility by *reasonably* discontinuous functions.

We propose to weaken the second fundamental concept at stake namely, functionality:

- We want to consider reducibility by *relatively continuous relations*.

Reductions

Fix sets X, Y, and subsets $A \subseteq X$, $B \subseteq Y$.
A *reduction* of A to B is a function $f : X \to Y$ such that

$$\forall x \in X \ (x \in A \leftrightarrow f(x) \in B).$$

A *total relation* from X to Y is a relation $R \subseteq X \times Y$ with $\forall x \in X \ \exists y \in Y \ R(x, y)$, in symbols $R : X \rightrightarrows Y$.

Definition

A *reduction* of A to B is a total relation $R : X \rightrightarrows Y$ such that

$$\forall x \in X \ \forall y \in Y \ \Big(R(x, y) \to (x \in A \leftrightarrow y \in B)\Big),$$

or equivalently

$$\forall x \in X \ \Big(x \in A \wedge R(x) \subseteq B\Big) \vee \Big(x \notin A \wedge R(x) \cap B = \emptyset\Big)$$

where $R(x) = \{y \in Y : R(x, y)\}$.

Reductions, basic properties

Basic Properties

Let $A \subseteq X$, $B \subseteq Y$, $C \subseteq Z$, and $R : X \rightrightarrows Y$, $T : Y \rightrightarrows Z$:

- *If R reduces A to B and T reduces B to C, then*

  $$T \circ R = \{(x, z) : \exists y \in Y \; R(x, y) \wedge T(y, z)\}$$

  reduces A to C.

Let $\mathcal{R}$ be a class of total relations from X to X with

1. the identity on X belongs $\mathcal{R}$,
2. $\mathcal{R}$ is closed under composition.

For $A, B \subseteq X$,

$$A \; \mathcal{R}\text{-reducible to } B \quad \longleftrightarrow \quad \exists R \in \mathcal{R} \quad R \text{ reduces } A \text{ to } B$$

This defines a quasi-order $\leq_{\mathcal{R}}$ on subsets of X.

Reductions, basic properties

Basic Properties

Let $A \subseteq X$, $B \subseteq Y$, $R, S : X \rightrightarrows Y$:

- *If $R \subseteq S$ and S reduces A to B, then R also reduces A to B.*

Let $\mathcal{R}$ be a class of total relations from X to X with

1. the identity on X belongs $\mathcal{R}$,
2. $\mathcal{R}$ is closed under composition.

Let $\overline{\mathcal{R}} = \{S : X \rightrightarrows X : \exists R \in \mathcal{R} \quad R \subseteq S\}$, then for any $A, B \subseteq X$,

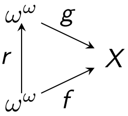
$$A\ \mathcal{R}\text{-reducible to } B \quad \longleftrightarrow \quad A\ \overline{\mathcal{R}}\text{-reducible to } B$$

In particular,

$$A \leq_W B \quad \longleftrightarrow \quad A\ \overline{\mathcal{W}}\text{-reduces to } B.$$

where $\mathcal{W} = \{\mathrm{graph}(f) : f : X \to X \text{ is continuous}\}$.

Admissible representations



Let $f, g :\subseteq \omega^\omega \to X$ be partial maps.
Say f *continuously reduces* to g, $f \leq_W g$, if

$$\exists \text{ continuous } r : \operatorname{dom} f \to \operatorname{dom} g \quad \forall \alpha \in \operatorname{dom} f \quad f(\alpha) = g \circ r(\alpha).$$

Proposition (Kreitz, Weihrauch, Schröder)

Let X be 2^{nd} countable T_0. There exists a partial map $\rho :\subseteq \omega^\omega \to X$ such that

- *ρ is continuous (and surjective),*
- *($\leq_W$-greatest) $\forall$ continuous $f :\subseteq \omega^\omega \to X$, $f \leq_W \rho$.*

Such a map is called an ***admissible representation*** *of X.*

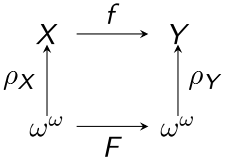
If $(V_n)_{n\in\omega}$ is a basis for X, then one can take $\rho :\subseteq \omega^\omega \to X$:

$$\rho(\alpha) = x \quad \longleftrightarrow \quad \{\alpha(k) : k \in \omega\} = \{n : x \in V_n\}.$$

Relatively continuous functions

Let X, Y be 2^{nd} countable T_0 spaces.
A map $f : X \to Y$ is *relatively continuous* if
for some (hence any) admissible representations ρ_X, ρ_Y
there exists a continuous $F : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ such that

$$\forall \alpha \in \operatorname{dom} \rho_X \quad f \circ \rho_X(\alpha) = \rho_Y \circ F(\alpha)$$



Proposition

Let X, Y be 2^{nd} countable T_0. A map $f : X \to Y$ is relatively continuous iff it is continuous.

If ρ_X is not injective, a continuous map $F : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ may very well induce *no* function from X to Y.
We can have $\alpha \neq \beta$, $\rho_X(\alpha) = \rho_X(\beta)$, and $\rho_Y(F(\alpha)) \neq \rho_Y(F(\beta))$.

Relatively continuous relations

Definition (Brattka, Hertling, Weihrauch)

X, Y 2^{nd} countable T_0 spaces.
A total relation $R : X \rightrightarrows Y$ is *relatively continuous* if for some (hence any) admissible representations ρ_X, ρ_Y there exists a continuous $F : \operatorname{dom} \rho_X \to \operatorname{dom} \rho_Y$ such that

$$\forall \alpha \in \operatorname{dom} \rho_X \quad R(\rho_X(\alpha), \rho_Y(F(\alpha)))$$

Basic Properties

1. *graphs of continuous functions are relatively continuous.*
2. *relatively continuous relations compose.*
3. *If $R, S : X \rightrightarrows Y$, R relatively continuous and $R \subseteq S$, then S is also relatively continuous.*

Reduction by relatively continuous relations

Definition

Let X be 2^{nd} countable T_0, $A, B \subseteq X$.
A *is reducible* to B, $A \preccurlyeq B$, if there exists a relatively continuous relation $R : X \rightrightarrows X$ that reduces A to B.

Basic Properties

1. *$\preccurlyeq$ is a quasi order on subsets of X.*
2. *If $A \leq_W B$, then $A \preccurlyeq B$.*
3. *For any admissible representation ρ of X, $A \preccurlyeq B$ iff there exists a continuous $F : \operatorname{dom} \rho \to \operatorname{dom} \rho$ with*

$$\forall \alpha \in \operatorname{dom} \rho \quad \left(\alpha \in \rho^{-1}(A) \quad \longleftrightarrow \quad F(\alpha) \in \rho^{-1}(B)\right).$$

the case of 0-dimensional spaces

Theorem

Let X be a 2^{nd} countable T_0 space. The following are equivalent.

1. *X is 0-dimensional.*
2. *X admits an injective admissible representation.*

So in a 2^{nd} countable 0-dim space X, for $R : X \rightrightarrows X$:

R is relatively continuous $\longleftrightarrow$ R admits a continuous uniformizing function.

This is not at all the case in the real line $\mathbb{R}$, for example.

Corollary

X 2^{nd} countable 0-dim, $A, B \subseteq X$: $A \leq_W B \leftrightarrow A \preccurlyeq B$.
That is, on 2^{nd} countable 0-dim spaces

Wedge reducibility = reducibility by relativ. cont. relations.

Borel representable spaces

Definition

A 2^{nd} countable T_0 space X is called *Borel representable space* if there exists an admissible representation ρ of X whose domain is Borel (in ω^ω).

Borel representable spaces include

- every Borel subspace of any Polish space, i.e. every Borel subspace of $[0,1]^\omega$.
- every Borel subspace of any quasi-Polish space, i.e. every Borel subspace of $\mathcal{P}(\omega)$ with the Scott topology.

Most (all?) properties of Wadge reducibility on 0-dim Polish spaces extends to arbitrary Borel representable spaces via the reducibility by relatively continuous relations.

The nice picture regained

Analysis of Wadge reducibility on ω^ω (Wadge, Martin, Monk) and Determinacy of Borel games (Martin) directly yield

Theorem

Let X be Borel representable.

1. *For Borel sets $A, B \subseteq X$, either $A \preccurlyeq B$ or $B \preccurlyeq A^\complement$ (so antichains have size at most 2).*
2. *$\preccurlyeq$ is well founded on Borel sets.*

And it follows by results of Saint Raymond and De Brecht that

Theorem

Let X be 2^{nd} countable T_0 and $\mathbf{\Gamma}$ be $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$ or $D_\theta(\mathbf{\Sigma}^0_\xi)$. Then if $B \in \mathbf{\Gamma}$ and $A \preccurlyeq B$, then $A \in \mathbf{\Gamma}$.