Pseudocompact inverse primitive (semi)topological semigroups

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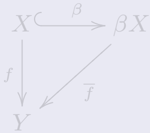
Čech-Stone Compactification

The Čech-Stone compactification of a Tychonoff space X is a compact Hausdorff space βX containing X as a dense subspace so that each continuous map $f\colon X \to Y$ to a compact Hausdorff space Y extends to a continuous map $\overline{f}\colon \beta X \to Y$, i.e., the following diagram commutes



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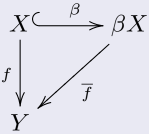

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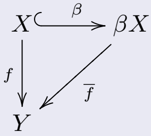

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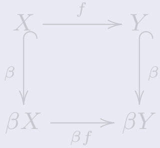

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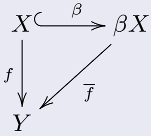

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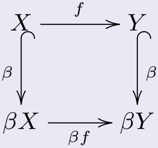

The Glicksberg Theorem, 1959

For Tychonoff topological spaces X, Y and Z a continuous map $f\colon X\times Y\to Z$ extends to a continuous map $\overline{f}\colon \beta X\times\beta Y\to\beta Z$ if and only if the product $X\times Y$ is pseudocompact.

A Tychonoff topological spaces X is called pseudocompact if every continuous map $f\colon X\to\mathbb{R}$ is bounded.

Reznichenko, 1994

Let X, Y and Z be Tychonoff topological spaces and continuous map $f\colon X\times Y\to Z$ be a continuous map. If X are Y pseudocompact then f extends to a separately continuous map $\overline{f}\colon \beta X\times\beta Y\to\beta Z$.

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Definition

Let X and Y are Tychonoff topological spaces. We shall say that (X, Y) is a Grothendieck pair if every continuous image of X in $C_p(Y)$ has the compact closure in $C_p(Y)$.

Arkhangel'skii, 1984

If (X, Y) is a Grothendieck pair then X is pseudocompact.

Reznichenko, 1994

Let X and Y be Tychonoff pseudocompact spaces. Then (X, Y) is a Grothendieck pair if and only if (Y, X) is a Grothendieck pair.

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If a Tychonoff pseudocompact space X satisfies one of the following conditions: (i) X is countably compact; (ii) X has countable tightness; (iii) X is separable; (iv) X is a k-space, then (X, Y) is a Grothendieck pair for every Tychonoff pseudocompact space Y.

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A topological space

- equipped with a continuous group operation and continuous inversion is called a *topological group*;
- equipped with a continuous group operation is called a *paratopological group*;
- equipped with a separately continuous group operation and continuous inversion is called a *quasitopological group*;
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A semigroup is a non-empty set with associative binary operation.

A semigroup S

- is called *inverse* if for every $x \in S$ there exists unique $y \in S$ such that $xyx = x$ and $yxy = y$, and in this case y is called inverse of x in S and denoted by x^{-1} (for an inverse semigroup S the map $\iota\colon S \to S$, $x \mapsto x^{-1}$ is called *inversion*);
- is a *semilattice* if it is a commutative semigroup of idempotents;
- is *simple* if S does not contain proper ideals;
- is *0-simple* if S does not contains no proper ideals distinct from $\{0\}$;
- is *completely simple* if S is simple and contains minimal left and right ideals;
- is *completely 0-simple* if S is 0-simple and contains minimal left and right ideals distinct from $\{0\}$.

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A subset idempotents $E(S)$ of a semigroup S admits a *natural partial order* $\leqslant$:

$$e \leqslant f \qquad \text{if and only if} \qquad ef = fe = e, \qquad e, f \in E(S).$$

An idempotent e of a semigroup S is *primitive* if it minimal in $E(S) \setminus \{0\}$.

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Later we shall assume that all spaces are Hausdorff.

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Let $\mathscr{C}$ be a class of compact Hausdorff semitopological semigroups. By a *$\mathscr{C}$-compactification* of a semitopological semigroup S we understand a pair $(\mathscr{C}(S), \eta)$ consisting of a compact semitopological semigroup $\mathscr{C}(S) \in \mathscr{C}$ and a continuous homomorphism $\eta : S \to \mathscr{C}(S)$ (called the *canonic homomorphism*) such that for each continuous homomorphism $h : S \to K$ to a semitopological semigroup $K \in \mathscr{C}$ there is a unique continuous homomorphism $\bar{h} : \mathscr{C}(S) \to K$ such that $h = \bar{h} \circ \eta$.

We shall be interested in $\mathscr{C}$-compactifications for the following classes of semigroups:

- $\mathscr{WAP}$ of compact semitopological semigroups;
- $\mathscr{AP}$ of compact topological semigroups.

The corresponding $\mathscr{C}$-compactifications of a semitopological semigroup S will be denoted by $\mathscr{WAP}(S)$ and $\mathscr{AP}(S)$. The notation came from the abbreviations for weakly almost periodic, almost periodic, and strongly almost periodic function rings that determine those compactifications.

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Reznichenko, 1995

For any Tychonoff countably compact semitopological semigroup S the semigroup operation of S extends to a separately continuous semigroup operation on βS, which implies that βS coincides with the $\mathscr{WAP}$-compactification of S.

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For any Tychonoff pseudocompact topological semigroup S the semigroup operation of S extends to a separately continuous semigroup operation βS, which implies that βS coincides with the $\mathscr{WAP}$-compactification of S.

Banakh, Dimitrova, 2010

For any Tychonoff topological semigroup S with pseudocompact square $S \times S$ the semigroup operation of S extends to a continuous semigroup operation on βS, which implies that βS coincides with the $\mathscr{AP}$-compactification of S.

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Comfort & Ross, 1966

The Tychonoff product of any non-empty family of pseudocompact topological groups is a pseudocompact space.

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The Čech-Stone compactification βG of a pseudocompact topological group is a topological group and the group operation of βG is an extension of the group operation of G.

Korovin, 1992

There exists a Tychonoff pseudocompact semitopological group which is not a paratopological group.

Hernandes & Tkachenko, 2006

There exist two Tychonoff pseudocompact quasitopological groups G and H such that the product $G \times H$ is not pseudocompact.

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The Čech-Stone compactification βG of a pseudocompact topological group is a topological group and the group operation of βG is an extension of the group operation of G.

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There exists a Tychonoff pseudocompact semitopological group which is not a paratopological group.

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There exist two Tychonoff pseudocompact quasitopological groups G and H such that the product $G \times H$ is not pseudocompact.

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Theorem

For a Tychonoff space X the following conditions are equivalent:

(i) X is pseudocompact;

(ii) every locally finite family of non-empty open subsets of X is finite;

(iii) every locally finite open cover of X has a finite subcover.

Ravsky, 2012

The Tychonoff product of any non-empty family of pseudocompact paratopological groups is a pseudocompact space.

G & Repovš, 2007

Let S be a 0-simple countable compact topological inverse semigroup. Then the Stone-Čech compactification of S admits a structure of 0-simple topological inverse semigroup with respect to which the inclusion mapping of S into βS is a topological isomorphism.

G & Pavlyk, 2013

Let $\{S_i : i \in \mathscr{I}\}$ be a non-empty family of primitive Hausdorff pseudocompact topological inverse semigroups. Then the direct product $\prod_{j\in\mathscr{J}} S_j$ with the Tychonoff topology is a pseudocompact topological inverse semigroup.

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Let S be a primitive pseudocompact topological inverse semigroup. Then the Stone-Čech compactification of S admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of S into βS is a topological isomorphism.

Comfort-Ross like Theorem

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Let S be a primitive pseudocompact topological inverse semigroup. Then the Stone-Čech compactification of S admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of S into βS is a topological isomorphism.

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Let $\{S_i : i \in \mathscr{J}\}$ be a non-empty family of primitive Hausdorff pseudocompact topological inverse semigroups. Then the direct product $\prod_{j \in \mathscr{J}} S_j$ with the Tychonoff topology is a pseudocompact topological inverse semigroup.

G & Pavlyk, 2013

Let S be a primitive pseudocompact topological inverse semigroup. Then the Stone-Čech compactification of S admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of S into βS is a topological isomorphism.

Clifford, 1951

Let S be a group and λ be a cardinal $\geqslant 1$. On the set $B_\lambda(S) = (\lambda \times S \times \lambda) \sqcup \{0\}$ we define the semigroup operation as follows

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & \text{if } \beta = \gamma; \\ 0, & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$, for all $\alpha, \beta, \gamma, \delta \in \lambda$ and $a, b \in S$. The semigroup $B_\lambda(S)$ is called the *Brandt semigroup*. Every completely 0-simple inverse semigroup is isomorphic to Brandt semigroup for some cardinal λ and group S.

For all $\alpha, \beta \in \lambda$ we denote $S_{\alpha,\beta} = \{(\alpha, s, \beta) \colon s \in S\}$.

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For all $\alpha, \beta \in \lambda$ we denote $S_{\alpha,\beta} = \{(\alpha, s, \beta) \colon s \in S\}$.

Let $\{S_\iota \colon \iota \in \mathscr{I}\}$ be a disjoint family of semigroups with zero such that 0_ι is zero in S_ι for any $\iota \in \mathscr{I}$. We put $S = \{0\} \cup \bigcup\{S_\iota \setminus \{0_\iota\} \colon \iota \in \mathscr{I}\}$, where $0 \notin \bigcup\{S_\iota \setminus \{0_\iota\} \colon \iota \in \mathscr{I}\}$, and define a semigroup operation on S in the following way

$$s \cdot t = \begin{cases} st, & \text{if } st \in S_\iota \setminus \{0_\iota\} \text{ for some } \iota \in \mathscr{I}; \\ 0, & \text{otherwise.} \end{cases}$$

The semigroup S with such defined operation is called the *orthogonal sum* of of the family of semigroups $\{S_\iota \colon \iota \in \mathscr{I}\}$ and in this case we shall write $S = \sum_{\iota \in \mathscr{I}} S_\iota$.

Petrich, 1984

A semigroup S is a primitive inverse semigroup if and only if S is the orthogonal sum of a non-empty family of Brandt semigroups.

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A semigroup S is a primitive inverse semigroup if and only if S is the orthogonal sum of a non-empty family of Brandt semigroups.

G & Ravsky, 2013

Let S be a Hausdorff primitive inverse countably compact semitopological semigroup and S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\colon i \in \mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Suppose that for every $i \in \mathscr{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i \in \lambda_i$, such that at least the one of the following conditions holds:

(1) the group $(G_i)_{\alpha_i,\alpha_i}$ is left precompact;

(2) $(G_i)_{\alpha_i,\alpha_i}$ is a pseudocompact paratopological group;

(3) the group $(G_i)_{\alpha_i,\alpha_i}$ is left ω-precompact pseudocompact;

(4) the subsemigroup $S_{\alpha_i,\alpha_i} = (G_i)_{\alpha_i,\alpha_i} \cup \{0\}$ is a topological semigroup.

Then S admits the unique topology which turns S into a semitopological semigroup.

We recall that a group G endowed with a topology is *left* (ω-)*precompact*, if for each neighborhood U of the unit of G there exists a (countable) finite subset F of G such that $FU = G$.

G & Ravsky, 2013

Let S be a semiregular primitive inverse pseudocompact semitopological semigroup and S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i)\colon i \in \mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Let for every $i \in \mathscr{I}$ there exists a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i \in \lambda_i$, such that at least the one of the following conditions holds:

(1) the group $(G_i)_{\alpha_i,\alpha_i}$ is left precompact;

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G & Ravsky, 2013

Let S be a Hausdorff primitive inverse pseudocompact topological semigroup and S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) \colon i \in \mathscr{I}\}$ of topological Brandt semigroups with zeros. Then the following assertions hold:

(i) every cardinal λ_i is finite;

(ii) every maximal subgroup of S is open-and-closed subset of S and hence is pseudocompact;

(iii) for every $i \in \mathscr{I}$ the maximal Brandt semigroup $B_{\lambda_i}(G_i)$ is a pseudocompact;

(iv) if $\mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$ is a base of the topology at the unity $(\alpha_i, e_i, \alpha_i)$ of a maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$ of S, $i \in \mathscr{I}$, such that $U \subseteq (G_i)_{\alpha_i,\alpha_i}$ for any $U \in \mathscr{B}_{(\alpha_i,e_i,\alpha_i)}$, then the family

$$\mathscr{B}_{(\beta_i,x,\gamma_i)} = \left\{(\beta_i, x, \alpha_i) \cdot U \cdot (\alpha_i, e_i, \gamma_i) \colon U \in \mathscr{B}_{(\alpha_i,e_i,\alpha_i)}\right\}$$

is a base of the topology at the point $(\beta_i, x, \gamma_i) \in (G_i)_{\beta_i,\gamma_i} \subseteq B_{\lambda_i}(G_i)$, for all $\beta_i, \gamma_i \in \lambda_i$;

if in addition the topological space S is semiregular then

(v) the family

$$\mathscr{B}_0 = \big\{S \backslash \big((G_{i_1})_{\alpha_{i_1},\beta_{i_1}} \cup \cdots \cup (G_{i_k})_{\alpha_{i_k},\beta_{i_k}}\big) \colon i_1, \ldots, i_k \in \mathscr{I}, \alpha_{i_k}, \beta_{i_k} \in \lambda_{i_k}, \; k \in \mathbb{N}, \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_k}, \beta_{i_k})\} \text{ is finite}\big\}$$

is a base of the topology at zero of S.

G & Ravsky, 2013

Let $\{S_j : j \in \mathscr{J}\}$ be a non-empty family of primitive semitopological inverse semigroups such that for each $j \in \mathscr{J}$ the semigroup S_j is either semiregular pseudocompact or Hausdorff countably compact, and moreover each maximal subgroup of S_j a pseudocompact paratopological group. Then the direct product $\prod_{j\in\mathscr{J}} S_j$ with the Tychonoff topology is a pseudocompact semitopological inverse semigroup.

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Let $\{S_i : i \in \mathscr{J}\}$ be a non-empty family of primitive inverse semiregular pseudocompact (Hausdorff countably) topological semigroups. Then the direct product $\prod_{j\in\mathscr{J}} S_j$ with the Tychonoff topology is a pseudocompact inverse topological semigroup.

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G & Ravsky, 2013

Let S be a primitive inverse pseudocompact quasi-regular topological semigroup. Then the Stone-Čech compactification of S admits a structure of primitive topological inverse semigroup with respect to which the inclusion mapping of S into βS is a topological isomorphism.

G & Ravsky, 2013

Let S be a regular primitive inverse countably compact semitopological semigroup and S be an orthogonal sum of the family $\{B_{\lambda_i}(G_i) : i \in \mathscr{I}\}$ of semitopological Brandt semigroups with zeros. Suppose that for every $i \in \mathscr{I}$ there exists s maximal non-zero subgroup $(G_i)_{\alpha_i,\alpha_i}$, $\alpha_i \in \lambda_i$, such that at least the one of the following conditions holds:

(1) the group $(G_i)_{\alpha_i,\alpha_i}$ is left precompact;

(2) the group $(G_i)_{\alpha_i,\alpha_i}$ is left ω-precompact pseudocompact;

(3) the subsemigroup $S_{\alpha_i,\alpha_i} = (G_i)_{\alpha_i,\alpha_i} \cup \{0\}$ is a topological semigroup.

Then the Stone-Čech compactification of S admits a structure of primitive inverse semitopological semigroup with continuous inversion with respect to which the inclusion mapping of S into βS is a topological isomorphism.

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Then the Stone-Čech compactification of S admits a structure of primitive inverse semitopological semigroup with continuous inversion with respect to which the inclusion mapping of S into βS is a topological isomorphism.

Thank You for Your attention!