A boolean algebraic approach to semiproper iterations

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The first part of this tutorial presents iterated forcing by means of boolean algebras focusing on semiproper iterations.
The second part of this tutorial (if it will exist) will apply these results to the analysis of category forcings and eventually on the use of category forcings to get generic absoluteness results.

The theory of semiproper forcing has been introduced by Shelah, in order to iterate with forcing that change cofinalities (Prikry forcing, Namba forcing,...).

In my opinion (I'm sure I'm not alone) his account of these results is unreadable.

1.1 Definition. We define and prove the following (A), (B), (C), (D), Def. 1.2 and claims 1.3(1), 1.4, by *simultaneous induction on* α (also for generic extensions of V):

(A) $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is an RCS iteration (RCS stands for revised countable support).

(B) a $\bar{Q}$-named ordinal (or $[j, \alpha)$-ordinal), (above a condition r).

(C) a $\bar{Q}$-named condition (or $[j, \alpha)$-condition), and we define $q{\restriction}\xi$, $q{\restriction}\{\xi\}$ for a $\bar{Q}$-named $[j, \alpha)$-condition $\underset{\sim}{q}$ and ordinal ξ and they are a member of P_ξ and a P_ξ-name of a member of $\hat{Q}_\xi$ respectively; of course $\xi \in [j, \alpha]$ (and $\xi \in [j, \alpha)$ respectively).

(D) the RCS-limit of $\bar{Q}$, $\mathrm{Rlim}\,\bar{Q}$ which satisfies $P_i \lessdot \mathrm{Rlim}\,\bar{Q}$ for every $i < \alpha$ and $p{\restriction}\xi$, $p{\restriction}\{\xi\}$ for $\xi < \alpha$, $p \in \mathrm{Rlim}\,\bar{Q}$.

(A) We define "$\bar{Q}$ is an RCS iteration"

$\alpha = 0$: no condition.

α is limit: $\bar{Q} = \langle P_i, Q_i : i < \alpha \rangle$ is an RCS iteration iff for every $\beta < \alpha$, $\bar{Q}{\restriction}\beta$ is one.

$\alpha = \beta + 1$: $\bar{Q}$ is an RCS iteration iff $\bar{Q}{\restriction}\beta$ is one, $P_\beta = \mathrm{Rlim}\,(\bar{Q}{\restriction}\beta)$ and Q_β is a P_β-name of a forcing notion.

(B) We define "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal of depth Υ above r" by induction on the ordinal Υ (and $\alpha = \ell g \bar{Q}$).

The intended meaning is an $(\mathrm{Rlim}\,\bar{Q})$-name of an ordinal of a special kind, however $\mathrm{Rlim}\,\bar{Q}$ is still not defined. So we use the part already known.

For $\Upsilon = 0$: "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal of depth Υ above r" means ζ is a (plain) ordinal in $[j, \alpha)$, i.e., $j \leq \zeta < \alpha, r \in P_{\zeta+1}$; but if $\zeta + 1 = \alpha$ then $r \in P_\zeta$.

For $\Upsilon > 0$: "$\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal of depth Υ above r" means that for some $\beta < \alpha$, (letting $\gamma = \beta + 1$ if $\beta + 1 < \alpha$ and $\gamma = \beta$ otherwise) $r \in P_\gamma$, and for some antichain $\mathcal{I}$ of P_γ, pre-dense above r, $\mathcal{I} = \{p_i : i < i_0\} \subseteq P_\gamma$, $\{\Upsilon_i : i < i_0\}$ and $\{\underset{\sim}{\zeta}_i : i < i_0\}$, we have $P_\gamma \vDash$ "$(r{\restriction}\gamma) \leq p_i$" (for simplicity), $\underset{\sim}{\zeta}_i$ is a $\bar{Q}$-named $[\max\{j, \beta\}, \alpha)$-ordinal of depth Υ_i above p_i, $\Upsilon_i < \Upsilon$, and $\underset{\sim}{\zeta}$ is $\underset{\sim}{\zeta}_i$ if

p_i and r (i.e., if p_i, r will be in the generic set then ζ will be ζ_i; this is informal but clear, see formal version in 1.2(1)).

Without Υ : We say ζ is a $\bar{Q}$-named $[j,\alpha)$-ordinal above r, if it is such for some depth.

Without $r : r = \emptyset$.

Similarly, we omit "$[j,\alpha) -$ " when $j = 0$.

(C) We define "q is a $\bar{Q}$-named $[j,\alpha)$-condition of depth Υ above r" and also $q\restriction\{\xi\}, q\restriction\xi$ and the $\bar{Q}$-named $[j,\alpha)$-ordinal $\zeta(q)$ associated with q.

The definition is similar to (B).

For $\Upsilon = 0$: We say "q is a $\bar{Q}$-named $[j,\alpha)$-condition of depth Υ above r" if for some ordinal $\zeta, j \le \zeta < \alpha$ and q is a P_ζ-name of a member of $\hat{Q}_\zeta$ (see 1.0(6)), $r \in P_{\zeta+1}$ but if $\zeta + 1 = \alpha$ then $r \in P_\zeta$ and for simplicity q is above $r\restriction\{\zeta\}$ i.e. if $\zeta + 1 < \alpha$ then $r\restriction\zeta \Vdash_{P_\zeta}$ " in $\hat{Q}_\zeta, r\restriction\{\zeta\} \le q$" (note: $r\restriction\zeta \in P_\zeta$, $r\restriction\{\zeta\}$ is a member of $\hat{Q}_\zeta$). We let

$$q\restriction\xi = \begin{cases} q & \text{if } \xi > \zeta + 1 \\ q & \text{if } \xi = \zeta + 1 \\ \emptyset_{P_\xi} & \text{if } \xi \le \zeta \end{cases}$$

$$q\restriction\{\xi\} = \begin{cases} q & \text{if } \xi = \zeta, \\ \emptyset_{Q_\xi} & \text{if } \xi \ne \zeta. \end{cases}$$

notes: $\emptyset \in P_0$ and remember 1.0(7). Finally we let $\zeta(q) = \zeta$. [What if we wave "q above $r\restriction\{\zeta\}$"? Then $\xi = \zeta + 1$ need special attention as in $\bar{Q}\restriction\xi$, r may not be in P_ζ so we have to transfer the information of q to "allowable" form, so $q\restriction\xi$ depend also on r; so q should also tell us who is r or require $r\restriction\zeta \Vdash [\hat{Q}_\zeta \models r\restriction\{\zeta\} \le q$" or we should write $q\restriction_r\xi$, $q\restriction_r\{\xi\}$.]

For $\Upsilon > 0$: We say q is a $\bar{Q}$-named $[j,\alpha)$-condition of depth Υ above r, if for some $\beta < \alpha$ (letting $\gamma = \beta + 1$ if $\beta + 1 < \alpha$ and $\gamma = \beta$ otherwise) for some $\bar{Q}$-named $[j,\alpha)$-ordinal of depth Υ above r, ζ, defined by β, γ, $\{p_i : i < i_0\} \subseteq P_\gamma, \{\Upsilon_i : i < i_0\}$, $\{\zeta_i : i < i_0\}$, we have for each $i < i_0$ a $\bar{Q}$-named $[\max\{\beta, j\}, \alpha)$-condition q_i of depth Υ_i above $r \bigcup p_i$ (see clause (c) in (D) below), so informally $\zeta(\underset{\sim}{q}_i) = \zeta_i$, and $\underset{\sim}{q}$ is $\underset{\sim}{q}_i$ if p_i and r are in the generic set of P_γ).

We then let $\zeta(\underset{\sim}{q}) = \zeta$.

Now we define $\underset{\sim}{q}\restriction\xi$ and $\underset{\sim}{q}\restriction\{\xi\}$; [really, we can just replace $\underset{\sim}{q}_i$ by $\underset{\sim}{q}_i\restriction\xi$, $\underset{\sim}{q}_i\restriction\{\xi\}$ respectively. In order to be pedantic, we need the following]. We define $\underset{\sim}{q}\restriction\xi$ as follows (below we ask $r \in \bigcup_{\epsilon<\xi} P_{\epsilon+1}$, because if ξ is a successor, $r \in P_\xi$ is a reasonable situation, if ξ a limit ordinal - not). *If* $r \in \bigcup_{\epsilon<\xi} P_{\epsilon+1}$ and $\beta+1<\xi$, then $\underset{\sim}{q}\restriction\xi$ is defined like $\underset{\sim}{q}$ replacing $\underset{\sim}{q}_i$ by $\underset{\sim}{q}_i\restriction\xi$. If $r \in \bigcup_{\epsilon<\xi} P_{\epsilon+1}$, $\beta+1=\xi=\alpha$, then $\underset{\sim}{q}\restriction\xi$ is $\underset{\sim}{q}$. If $r \in \bigcup_{\epsilon<\xi} P_{\epsilon+1}$, $\beta+1=\xi<\alpha$ then $\underset{\sim}{q}\restriction\xi$ is the following P_β-name of a member of $\hat{\underset{\sim}{Q}}_\beta$:

if $r\restriction\beta \in \underset{\sim}{G}_{P_\beta}$ then $\underset{\sim}{q}\restriction\xi$ is $\{(p_i\restriction\{\beta\}, \underset{\sim}{q}_i) : p_i\restriction\beta \in \underset{\sim}{G}_{P_\beta}, i<i_0\} \in \hat{\underset{\sim}{Q}}$.

If $r \in \bigcup_{\epsilon<\xi} P_{\epsilon+1}$, $\beta+1>\xi$ or $r \notin \bigcup_{\epsilon<\xi} P_{\epsilon+1}$ *then*: $\underset{\sim}{q}\restriction\xi$ is $\emptyset$ (or not defined).

Similarly for $\underset{\sim}{q}\restriction\{\xi\}$. If $r \in P_{\xi+1}$ (or $r \in P_\xi$), $\gamma \leq \xi$ then $\underset{\sim}{q}\restriction\{\xi\}$ is defined like $\underset{\sim}{q}$ replacing $\underset{\sim}{q}_i$ by $\underset{\sim}{q}_i\restriction\{\xi\}$. If $r \in P_{\xi+1}$, $\beta<\gamma=\xi+1$ (hence $\beta=\xi<\alpha$) then $\underset{\sim}{q}\restriction\{\xi\}$ is the following P_β-name of a member of $\hat{\underset{\sim}{Q}}_\beta$: $\{(r\restriction\{\beta\} \cup p_i\restriction\{\beta\}, \underset{\sim}{q}_i\restriction\{\beta\}) : p_i\restriction\beta \in \underset{\sim}{G}_{P_\beta}$ and $r\restriction\beta \in P_\beta$ and $i<i_0\}$. If $r \in P_{\xi+1}$, $\beta=\gamma=\xi+1$ (actually is ruled out) or $\gamma>\xi+1$ then $\underset{\sim}{q}\restriction\{\xi\}$ is $\emptyset$. If $r \notin P_{\xi+1}$, then $\underset{\sim}{q}\restriction\{\xi\}$ is $\emptyset$ (or not defined).

[The definitions of $\zeta(\underset{\sim}{q}\restriction\xi), \zeta(\underset{\sim}{q}\restriction\{\xi\})$ are left to the reader].

We omit Υ and/or "$[j,\alpha)-$" if this holds for some ordinal Υ and/or $j=0$. We omit r when $r=\emptyset(=\emptyset_{P_0})$. We leave the definition of $\underset{\sim}{q}\restriction[\zeta,\xi)$ to the reader.

(D) We define $\mathrm{Rlim}\bar{Q}$ as follows:

if $\alpha=0$: $\mathrm{Rlim}\bar{Q}$ is trivial forcing with just one condition: $\emptyset=\emptyset_{P_0}$;

if $\alpha>0$: we call $\underset{\sim}{q}$ an atomic condition of *Rlim*$\bar{Q}$, if it is a $\bar{Q}$-named condition.

The set of conditions in $P_\alpha = \mathrm{Rlim}\bar{Q}$ is

$\{p : p$ a countable set of atomic conditions; and for every $\beta<\alpha$, $p\restriction\beta \overset{\mathrm{def}}{=} \{r\restriction\beta : r \in p\} \in P_\beta$, and $p\restriction\beta \Vdash_{P_\beta}$ "$p\restriction\{\beta\} \overset{\mathrm{def}}{=} \{r\restriction\{\beta\} : r \in p\}$ has an upper bound in $\hat{\underset{\sim}{Q}}_\beta$"$\}$.

The order is inclusion, (but in later sections we sometimes ignore the difference between $p \le q$ and $p \Vdash$ "$q \in \underset{\sim}{G}$")

Now we have to show:

(a) $P_\beta \lessdot \mathrm{Rlim}\bar{Q}$ (for $\beta < \alpha$). [By 1.4(1) below.]

(b) For $\beta < \alpha$, any $(\bar{Q} \restriction \beta)$-named $[j, \beta)$-ordinal (or condition) above r is a $\bar{Q}$-named $[j, \alpha)$-ordinal (or condition) above r. [Why? Obvious.]

(c) If $\xi < \alpha$, $\underset{\sim}{q}$ is a $\bar{Q}$-named (atomic) condition above r, $r \in \bigcup_{\varepsilon<\xi} P_\varepsilon$, then $\underset{\sim}{q} \restriction \xi$ is a $(\bar{Q} \restriction \xi)$-named (atomic) condition above r. [Why? Obvious.]

(d) If $\beta_1 < \beta_2 < \alpha$, $p \in P_{\beta_2} \setminus P_{\beta_1}$, $p \le q$ in P_{β_2} then $q \notin P_{\beta_1}$ (though it may be equivalent to one).

(e) If $\xi < \alpha$, $\underset{\sim}{q}$ a $\bar{Q}$-named atomic condition above r, $r \in \bigcup\limits_{\varepsilon<\xi} P_\xi$ *then* $\Vdash_P$ "$\underset{\sim}{q} \restriction \{\xi\}$ is a member of $\underset{\sim}{\hat{Q}}_\xi$ ".

1.1A Explanation. 1)What will occur if we simplify by letting in 1.1(B), for $\Upsilon > 0$, $\gamma = \beta$ always? Nothing happens, except that 1.5(3) is no longer true; though this is used later, we can manage without it too, though less esthetically; for variety, XIV 2.6 = [Sh:250, 2.6] is developed in this way (for a generalization called κ-RS, our case is $\kappa = \aleph_1$). For the case which interests us the two definitions are equivalent - by the proof of 2.6 (here).

2) So why in 1.1(B), for $\Upsilon > 0$, we do not let $\gamma = \beta + 1$ always? If $\beta + 1 = \alpha$, we fall into a vicious circle; defining $P_{\beta+1}$ using conditions in $P_{\beta+1}$; alternatively see XIV §1.

* * *

1.1B Remark. We can obviously define $\bar{Q}$-named sets; but for conditions (and ordinals for them) we want to avoid the vicious circle of using names which are interpreted only after forcing with them below.

1.2 Definition.

(1) Suppose $\underset{\sim}{\zeta}$ is a $\bar{Q}$-named $[j, \alpha)$-ordinal above r, $r \in G \subseteq \bigcup_{i<\alpha} P_i$ and $G \cap P_i$ generic over V (whenever $i < \alpha$) (say G is in some generic extension of V).

Donder and Fuchs gave another account of these results introducing iterated forcing by means of directed systems of complete boolean algebras.

A satisfactory version of their approach to iterated forcing has never been published and also their drafts haven't circulated that much.

- A first aim of this tutorial is to fill the gap in the literature about this part of Shelah's work and make all of us comfortable with the use of revised countable support iterations.
- A second aim of this tutorial is to show that we should refocus much more our treatment of forcing basing it on the boolean valued models approach.
  In my eyes many problems become much easier to handle if we use complete boolean algebras instead of posets.
  Especially problems dealing with general questions about forcing rather than with specific proofs of specific consistency results obtained with forcing.

A full account of the results on iterations and semiproperness I will sketch here can be downloaded

http://www.personalweb.unito.it/matteo.viale/semiproperforcing.pdf.

It is the outcome of a PhD course I gave on these matters last spring and fall.

These notes developed on the master thesis of Fiorella Guichardaz and were written by me, Giorgio Audrito and Silvia Steila, Raphael Carroy also contributed nice ideas on how to get a topological characterization of (semi)properness.

Definition

A *poset* (partially ordered set) is a set P together with a binary relation $\leq$ on P which is transitive, reflexive and antisymmetric.

- Given $a, b \in P$, $a \perp b$ ($a \parallel b$) denote the incompatibility and compatibility relation.
- For a given $A \subset P$,

  $$\downarrow A = \{p : \exists q \in A, p \leq_P q\},$$

  and

  $$\uparrow A = \{p : \exists q \in A, p \geq_P q\}.$$
- A set $B \subset P$ is predense iff $\downarrow B$ is dense.
- A poset P is separative iff for all $p \not\leq q \in P$, there exists $r \in P$ with $r \leq p$, $r \perp q$.
- A poset P is $< \lambda$-cc (chain condition) iff $|A| < \lambda$ for all maximal antichains $A \subset P$.

Definition

A set $I \subset P$ is an *ideal* in P iff it is downward closed and upward directed.
A set $F \subset P$ is a *filter* in P iff it is upward closed and downward directed.

Definition

Let M be a model of ZFC and $P \in M$ be a poset. A filter $G \subset P$ is M-generic for P if and only if it intersects every dense set D of P in M. Equivalently, a filter G is M-generic if it intersects every maximal antichain.

Fact

If $P \in M$ is a separative poset, no filter in M is M-generic.

Definition

A poset P is a *lattice* if any two elements a, b have a unique supremum $a \vee b$ (least upper bound, join) and infimum $a \wedge b$ (greatest lower bound, meet).

- A lattice is *distributive* if the operations of join and meet distribute over each other.
- A lattice $\mathbb{L}$ is *bounded* if it has a least element ($0_{\mathbb{L}}$) and a greatest element ($1_{\mathbb{L}}$).
- A lattice is *complemented* if it is bounded lattice and every element a has a complement denoted $\neg a$ satisfying $a \vee \neg a = 1$ and $a \wedge \neg a = 0$.
- A *boolean algebra* is a complemented distributive lattice. A boolean algebra is *complete* iff every subset has a supremum and an infimum.

Fact

Given a boolean algebra $\mathbb{B}$, $\mathbb{B} \setminus \{0\} = \mathbb{B}^+$ is a separative poset with the order relation $a \leq_{\mathbb{B}^+} b$ given by any of the following requirement on a and b:

- $a \wedge b = a$,
- $a \vee b = b$.

To any poset we can associate a unique (up to isomorphism) boolean completion (and its associated Stone space):

Theorem

For every poset P there exists a unique (up to isomorphism) complete boolean algebra $\mathbb{B}$ (the boolean completion of P) with a dense embedding $i_P : P \to \mathbb{B}^+$ such that for any $p, q \in P$:

- $p \leq q \implies i_P(p) \leq i_P(q)$,
- $p \perp q \implies i_P(p) \perp i_P(q)$,
- $i_P[P]$ *is a dense subset of* $\mathbb{B}^+$.

Definition

Let V be a transitive model of ZFC and $\mathbb{B}$ be a complete boolean algebra in V. A set $U \subset \mathbb{B}$ is an *ultrafilter* in $\mathbb{B}$ if and only if U is a filter and for any $b \in \mathbb{B}$, $b \in U$ or $\neg b \in U$.

If I is an ideal of $\mathbb{B}$, the *quotient* $\mathbb{B}/I$ is the quotient of $\mathbb{B}$ with respect to the equivalence relation defined by $a \approx b \Leftrightarrow a \triangle b \in I$
$(a \triangle b = (a \vee b) \wedge \neg(a \wedge b))$.
$\mathbb{B}/I$ is always a boolean algebra (but in general not complete).

Notation

Let $\mathbb{B}$ be a complete boolean algebra. $X \subset \mathbb{B}$ is a pre-filter if $\uparrow X$ is a filter and is a pre-ideal if $\downarrow X$ is an ideal.
Given $X \subset \mathbb{B}$, we let $I^* = \{\neg a : a \in I\}$. It is well known that $I^{**} = I$ and I is an ideal iff I^* is a filter.
With an abuse of notation, if G is a pre-filter on $\mathbb{B}$, we write $\mathbb{B}/G$ also to denote $\mathbb{B}/(\uparrow G)^*$.

Definition

Let V be a transitive model of ZFC and $\mathbb{B}$ be a complete boolean algebra in V.

$$V^{\mathbb{B}} = \{\dot{a} \in V : \dot{a} : V^{\mathbb{B}} \to \mathbb{B} \text{ is a function }\}.$$

We let for the atomic formulas $x \in y$, $x \subseteq y$, $x = y$:

- $\left[\!\left[\dot{b}_0 \in \dot{b}_1\right]\!\right]_{\mathbb{B}} = \bigvee \left\{\left[\!\left[\dot{a} = \dot{b}_0\right]\!\right]_{\mathbb{B}} \wedge \dot{b}_0(\dot{a}) : \dot{a} \in \mathrm{dom}(\dot{b}_1)\right\}$,
- $\left[\!\left[\dot{b}_0 \subseteq \dot{b}_1\right]\!\right]_{\mathbb{B}} = \bigwedge \left\{\neg \dot{b}_0(\dot{a}) \vee \left[\!\left[\dot{a} \in \dot{b}_0\right]\!\right]_{\mathbb{B}} : \dot{a} \in \mathrm{dom}(\dot{b}_0)\right\}$,
- $\left[\!\left[\dot{b}_0 = \dot{b}_1\right]\!\right]_{\mathbb{B}} = \left[\!\left[\dot{b}_0 \subseteq \dot{b}_1\right]\!\right]_{\mathbb{B}} \wedge \left[\!\left[\dot{b}_1 \subseteq \dot{b}_0\right]\!\right]_{\mathbb{B}}$.

For general formulas $\phi(x_0, \ldots, x_n)$, we let:

- $[\![\neg\phi]\!]_{\mathbb{B}} = \neg [\![\phi]\!]_{\mathbb{B}}$,
- $[\![\phi \wedge \psi]\!]_{\mathbb{B}} = [\![\phi]\!]_{\mathbb{B}} \wedge [\![\psi]\!]_{\mathbb{B}}$,
- $[\![\phi \vee \psi]\!]_{\mathbb{B}} = [\![\phi]\!]_{\mathbb{B}} \vee [\![\psi]\!]_{\mathbb{B}}$,
- $\left[\!\left[\exists x \phi(x, \dot{b}_1, \ldots, \dot{b}_n)\right]\!\right]_{\mathbb{B}} = \bigvee \left\{\left[\!\left[\phi(\dot{a}, \dot{b}_1, \ldots, \dot{b}_n)\right]\!\right]_{\mathbb{B}} : \dot{a} \in V^{\mathbb{B}}\right\}$.

When the context is clear, we will omit the index.

Notation

For a complete boolean algebra $\mathbb{B}$, $\dot{G}_{\mathbb{B}} \in V^{\mathbb{B}}$ always denote the canonical name for a V-generic filter for $\mathbb{B}$, i.e.

$$\dot{G}_{\mathbb{B}} = \{\langle \check{b}, b \rangle : b \in \mathbb{B}\}.$$

Boolean algebras allows to make sense of forcing without appealing to the existence of generic filters.

Theorem (Boolean valued forcing theorem)

Let V be a transitive model of ZFC *and* $\mathbb{B}$ *be a complete boolean algebra in* V*. Let* G *be any ultrafilter on* $\mathbb{B}$*. For* $\dot{a}, \dot{b} \in V^{\mathbb{B}}$ *we let*

- $\dot{b} =_G \dot{a}$ *iff* $[\![\dot{b} = \dot{a}]\!] \in G$,
- $[\dot{b}]_G = \{\dot{a} : \dot{b} =_G \dot{a}\}$,
- $[\dot{b}]_G \in_G [\dot{a}]_G$ *iff* $[\![\dot{b} \in \dot{a}]\!] \in G$,
- $V^{\mathbb{B}}/G = \{[\dot{b}]_G : \dot{b} \in V^{\mathbb{B}}\}$.

Then:

1. $(V^{\mathbb{B}}/G, \in_G)$ *is a model of* ZFC
2. $(V^{\mathbb{B}}/G, \in_G)$ *models* $\phi([\dot{b}_1]_G, \ldots, [\dot{b}_n]_G)$ *iff* $[\![\phi(\dot{b}_1, \ldots, \dot{b}_n)]\!] \in G$.

Theorem (Cohen's forcing theorem)

Let V be transitive model of ZFC, $\mathbb{B} \in V$ be a complete boolean algebra G be a V-generic filter for $\mathbb{B}$. Then:

1. *$V[G]$ is isomorphic to $V^{\mathbb{B}}/G$ via the map which sends $\dot{b}_G$ to $[\dot{b}]_G$.*
2. *$V[G] \models \phi((\dot{b}_1)_G, \ldots, (\dot{b}_n)_G)$ iff $[\![\phi(\dot{b}_1, \ldots, \dot{b}_n)]\!] \in G$.*
3. *$b \leq_{\mathbb{B}} [\![\phi(\dot{b}_1, \ldots, \dot{b}_n)]\!]$ iff $V[G] \models \phi((\dot{b}_1)_G, \ldots, (\dot{b}_n)_G)$ for all V-generic filters G for $\mathbb{B}$ such that $b \in G$.*

Notation

Given a partial order P and $\dot{b}_1, \ldots, \dot{b}_n \in V^{\mathrm{RO}(P)}$ we say that $p \Vdash_P \phi(\dot{b}_1, \ldots, \dot{b}_n)$ iff $i_P(p) \leq [\![\phi(\dot{b}_1, \ldots, \dot{b}_n)]\!]$.

Lemma (Mixing)

Let $\mathbb{B}$ be a complete boolean algebra and $\{\dot{b}_a : a \in A\}$ be a family of $\mathbb{B}$-names indexed by an antichain. Then there exists $\dot{b} \in V^{\mathbb{B}}$ such that $[\![\dot{b} = \dot{b}_a]\!] \geq a$ for all $a \in A$.

Lemma (Fullness)

Let $\mathbb{B}$ be a complete boolean algebra. For all formula $\phi(x, x_1, \ldots, x_n)$ and $\dot{b}_1, \ldots, \dot{b}_n \in V^{\mathbb{B}}$, there is $\dot{b} \in V^{\mathbb{B}}$ such that

$$[\![\exists x \phi(x, \dot{b}_1, \ldots \dot{b}_n)]\!] = [\![\phi(\dot{b}, \dot{b}_1, \ldots \dot{b}_n)]\!] .$$

Fact

Let V be transitive model of ZFC*, $\mathbb{B} \in V$ be a complete boolean algebra, G be a V-generic ultrafilter for $\mathbb{B}$. Then $\bigwedge A \in G$ for any $A \subset G$ which belongs to V.*

Definition

Let $\mathbb{B}$, $\mathbb{C}$ be complete boolean algebras, $i : \mathbb{B} \to \mathbb{C}$ is a *complete homomorphism* iff it is an homomorphism that preserves arbitrary suprema. i is a *regular embedding* iff it is an injective complete homomorphism of boolean algebras.

Lemma

Let $i : \mathbb{B} \to \mathbb{C}$ be an homomorphism of boolean algebras. TFAE:

- *i is a complete homomorphism.*
- *For every V-generic filter G for $\mathbb{C}$, $H = i^{-1}[G]$ is a V-generic filter for $\mathbb{B}$ and $\mathbb{C}/i[H]$ is a complete boolean algebra in $V[H]$.*

Remark

Complete embeddings of posets give rise to regular embeddings.
To handle two step iterations complete homomorphisms suffice.
Regular homomorphisms are needed to handle iterations of limit length, not two step iterations!

From now on we shall focus just on regular homomorphisms since we want to deal with arbitrary iterations.

Definition

Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, the *retraction* associated to i is the map

$$\begin{array}{rccc} \pi_i : & \mathbb{C} & \to & \mathbb{B} \\ & c & \mapsto & \bigwedge \{b \in \mathbb{B} : \ i(b) \geq c\} \end{array}$$

Proposition

Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $b \in \mathbb{B}$, $c, d \in \mathbb{C}$ be arbitrary. Then,

1. *$\pi_i \circ i(b) = b$ hence π_i is surjective;*
2. *$i \circ \pi_i(c) \geq c$ hence π_i maps $\mathbb{C}^+$ to $\mathbb{B}^+$;*
3. *π_i preserves joins, i.e. $\pi_i(\bigvee X) = \bigvee \pi_i[X]$ for all $X \subseteq \mathbb{C}$;*
4. *$i(b) = \bigvee \{e : \pi_i(e) \leq b\}$.*
5. *$\pi_i(c \wedge i(b)) = \pi_i(c) \wedge b = \bigvee \{\pi_i(e) : e \leq c, \pi_i(e) \leq b\}$;*

Remark

If a regular embedding $i : \mathbb{B} \to \mathbb{C}$ is not surjective, π_i does not preserve neither meets nor complements, but $\pi_i(d \wedge c) \leq \pi_i(d) \wedge \pi_i(c)$ and $\pi_i(\neg c) \geq \neg \pi_i(c)$.

Lemma

Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $D \subset \mathbb{B}$, $E \subset \mathbb{C}$ be predense sets, then $i[D]$ and $\pi_i[E]$ are predense (i.e. predense subsets are mapped into predense subsets). Moreover π_i maps V-generic filter in V-generic filters.

Complete homomorphisms of complete boolean algebras extend to natural Δ_1-elementary maps between boolean valued models.

Proposition

Let $i : \mathbb{B} \to \mathbb{C}$ be a complete homomorphism, and define by recursion $\hat{i} : V^{\mathbb{B}} \to V^{\mathbb{C}}$ by

$$\hat{i}(\dot{b})(\hat{i}(\dot{a})) = i \circ \dot{b}(\dot{a})$$

for all $\dot{a} \in \mathrm{dom}(\dot{b}) \in V^{\mathbb{B}}$. Then the map $\hat{i}$ is Δ_1-elementary, i.e. for every Δ_1 formula ϕ,

$$i\left(\llbracket \phi(\dot{b}_1, \ldots, \dot{b}_n) \rrbracket_{\mathbb{B}}\right) = \left\llbracket \phi(\hat{i}(\dot{b}_1), \ldots, \hat{i}(\dot{b}_n)) \right\rrbracket_{\mathbb{C}}$$

In general for the sake of readability we shall confuse $\mathbb{B}$-names with their defining properties:

- If we have in V a collection $\{\dot{b}_i : i \in I\}$ of $\mathbb{B}$-names, we confuse $\{\dot{b}_i : i \in I\}$ with a $\mathbb{B}$-name $\dot{b}$ such that for all $\dot{a} \in V^{\mathbb{B}}$

$$\llbracket \dot{a} \in \dot{b} \rrbracket = \llbracket \exists i \in \check{I}\, \dot{a} = \dot{b}_i \rrbracket .$$

- If $i : \mathbb{B} \to \mathbb{C}$ is a complete homomorphism, we denote by $\mathbb{C}/i[\dot{G}]$ a $\mathbb{B}$-name $\dot{b}$ such that

$$\left\llbracket \dot{b} \text{ is the quotient of } \mathbb{C} \text{ modulo } i[\dot{G}_{\mathbb{B}}] \right\rrbracket = 1_{\mathbb{B}}.$$

Definition

Let $\mathbb{B}$ be a complete boolean algebra, and $\dot{\mathbb{C}}$ be a $\mathbb{B}$-name for a complete boolean algebra. $\mathbb{B} * \dot{\mathbb{C}}$ is the boolean algebra in V whose elements are the $\dot{a} \in V^{\mathbb{B}}$ such that $[\![\dot{a} \in \dot{\mathbb{C}}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}$ modulo the equivalence relation:

$$\dot{a} \approx \dot{b} \;\Leftrightarrow\; [\![\dot{a} = \dot{b}]\!]_{\mathbb{B}} = 1,$$

with the following operations:

$$[\dot{d}] \vee_{\mathbb{B}*\dot{\mathbb{C}}} [\dot{e}] = [\dot{f}] \iff \left[\!\!\left[\dot{d} \vee_{\dot{\mathbb{C}}} \dot{e} = \dot{f}\right]\!\!\right] = 1_{\mathbb{B}};$$

$$\neg_{\mathbb{B}*\dot{\mathbb{C}}}[\dot{d}] = [\dot{e}]$$

for any $\dot{e}$ such that $[\![\dot{e} = \neg_{\dot{\mathbb{C}}} \dot{d}]\!] = 1_{\mathbb{B}}$.

Literally speaking our definition of $\mathbb{B} * \dot{\mathbb{C}}$ yields an object whose domain is a family of proper classes.. By means of Scott's trick we can arrange so that $\mathbb{B} * \dot{\mathbb{C}}$ is indeed a set.

Lemma

Let $\mathbb{B}$ be a complete boolean algebra, and $\dot{\mathbb{C}}$ be a $\mathbb{B}$-name for a complete boolean algebra. Then $\mathbb{B} * \dot{\mathbb{C}}$ is a complete boolean algebra and the maps $i_{\mathbb{B},\dot{\mathbb{C}}}$, $\pi_{\mathbb{B}*\dot{\mathbb{C}}}$ defined as

$$\begin{array}{rccc} i_{\mathbb{B},\dot{\mathbb{C}}}: & \mathbb{B} & \to & \mathbb{B} * \dot{\mathbb{C}} \\ & b & \mapsto & [\dot{d}_b]_{\approx} \\ \pi_{\mathbb{B},\dot{\mathbb{C}}}: & \mathbb{B} * \dot{\mathbb{C}} & \to & \mathbb{B} \\ & [\dot{c}]_{\approx} & \mapsto & [\![\dot{c} > 0]\!] \end{array}$$

where $\dot{d}_b \in V^{\mathbb{B}}$ is a $\mathbb{B}$-name for an element of $\dot{\mathbb{C}}$ such that $[\![\dot{d}_b = 1_{\dot{\mathbb{C}}}]\!]_{\mathbb{B}} = b$ and $[\![\dot{d}_b = 0_{\dot{\mathbb{C}}}]\!]_{\mathbb{B}} = \neg b$, are a regular embedding with its associated retraction.

Fact

$A = \{[\dot{c}_\alpha]_\approx : \alpha \in \lambda\}$ is a maximal antichain in $\mathbb{D} = \mathbb{B} * \dot{\mathbb{C}}$, if and only if

$$\left[\!\left[\{\dot{c}_\alpha : \alpha \in \lambda\} \text{ is a maximal antichain in } \dot{\mathbb{C}}\right]\!\right] = 1.$$

Lemma

Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding, $\dot{G}_\mathbb{B}$ be the canonical name for a generic filter for $\mathbb{B}$ and $\dot{d}$ be a $\mathbb{B}$-name for an element of $\mathbb{C}/_{\dot{G}}$. Then there exists a unique $c \in \mathbb{C}$ such that $\left[\!\left[\dot{d} = [c]_{i[\dot{G}_\mathbb{B}]}\right]\!\right] = 1_\mathbb{B}$.

Proposition

Let $i : \mathbb{B} \to \mathbb{C}$ be a regular embedding of complete boolean algebras and G be a V-generic filter for $\mathbb{B}$. Then $\mathbb{C}/_G$, defined with abuse of notation as the quotient of $\mathbb{C}$ with the filter generated by $i[G]$, is a complete boolean algebra in $V[G]$.

Proposition

Let $\mathbb{B}$, $\mathbb{C}_0$, $\mathbb{C}_1$ be complete boolean algebras, and let G be a V-generic filter for $\mathbb{B}$. Let i_0, i_1, j form a commutative diagram of regular embeddings as in the following picture:

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "i_0"] \arrow[dr, "i_1"'] & \mathbb{C}_0 \arrow[d, "j"] \\
& \mathbb{C}_1
\end{tikzcd}$$

Then $j/_G : \mathbb{C}_0/_G \to \mathbb{C}_1/_G$ defined by $j/_G([c]_{i_0[G]}) = [j(c)]_{i_1[G]}$ is a well-defined regular embedding of complete boolean algebras in $V[G]$ with associated retraction π such that $\pi([c]_{i_1[G]}) = [\pi_j(c)]_{i_0[G]}$.

Theorem

If $i : \mathbb{B} \to \mathbb{C}$ *is a regular embedding of complete boolean algebra, then* $\mathbb{B} * (\mathbb{C}/_{i[\dot{G}_{\mathbb{B}}]}) \cong \mathbb{C}$.

Fact

Assume $\mathbb{B} \in V$ is a complete boolean algebra, $\dot{\mathbb{C}} \in V^{\mathbb{B}}$ is a $\mathbb{B}$-name for a complete boolean algebra and $\dot{\mathbb{D}} \in V^{\mathbb{B}*\dot{\mathbb{C}}}$ is a $\mathbb{B} * \dot{\mathbb{D}}$-name for a complete boolean algebra.
Let G be any ultrafilter on $\mathbb{B}$ and K be any ultrafilter on $\mathbb{B}/G$. Set

$$H = \{c : [c]_G \in K\}$$

Then

$$K = \{[c]_G : c \in H\}$$

and $((\mathbb{B} * \dot{\mathbb{C}}) * \dot{\mathbb{D}}/_G)/_K$ is isomorphic to $(\mathbb{B} * \dot{\mathbb{C}}) * \dot{\mathbb{D}}/_H$ via the map $[[c]_G]_K \mapsto [c]_H$.

Definition

$\mathcal{F} = \{i_{\alpha\beta} : \mathbb{B}_\alpha \to \mathbb{B}_\beta : \alpha \leq \beta < \lambda\}$ is a *complete iteration system* of complete boolean algebras iff for all $\alpha \leq \beta \leq \gamma < \lambda$:

1. $\mathbb{B}_\alpha$ is a complete boolean algebra and $i_{\alpha\alpha}$ is the identity on it;
2. $i_{\alpha\beta}$ is a regular embedding with associated retraction $\pi_{\alpha\beta}$;
3. $i_{\beta\gamma} \circ i_{\alpha\beta} = i_{\alpha\gamma}$.

If $\gamma < \lambda$, we define $\mathcal{F} \upharpoonright \gamma = \left\{i_{\alpha\beta} : \alpha \leq \beta < \gamma\right\}$.

Definition

Let $\mathcal{F}$ be a complete iteration system of length λ.

- The *inverse limit* of the iteration is

$$T(\mathcal{F}) = \left\{ f \in \prod_{\alpha<\lambda} \mathbb{B}_\alpha : \ \forall\alpha\forall\beta > \alpha \ \pi_{\alpha\beta}(f(\beta)) = f(\alpha) \right\}$$

  and its elements are called *threads*.

- The *direct limit* is

$$C(\mathcal{F}) = \left\{ f \in T(\mathcal{F}) : \ \exists\alpha\forall\beta > \alpha \ f(\beta) = i_{\alpha\beta}(f(\alpha)) \right\}$$

  and its elements are called *constant threads*. The support of a constant thread $\mathrm{supp}(f)$ is the least α such that $i_{\alpha\beta} \circ f(\alpha) = f(\beta)$ for all $\beta \geq \alpha$.

- The *revised countable support limit* is

$$RCS(\mathcal{F}) = \left\{ f \in T(\mathcal{F}) : \ f \in C(\mathcal{F}) \vee \exists\alpha \ f(\alpha) \Vdash_{\mathbb{B}_\alpha} \mathrm{cf}(\check{\lambda}) = \check{\omega} \right\}$$

Remark

$T(\mathcal{F})$ ordered by pairwise comparison of threads is a separative partial order and this ordering is inherited by $C(\mathcal{F})$ and $RCS(\mathcal{F})$.

We can define on $T(\mathcal{F})$ a natural join operation.

Definition

Let A be any subset of $T(\mathcal{F})$. We define the *pointwise supremum* of A as

$$\tilde{\bigvee} A = \langle \bigvee \{f(\alpha) : f \in A\} : \alpha < \lambda \rangle.$$

The previous definition makes sense since $\tilde{\bigvee} A$ is a thread because projections preserves suprema.

Remark

WATCH OUT!!! In general the pointwise supremum $\tilde{\bigvee} A$ of some $A \in T(\mathcal{F})$ is strictly larger than the supremum of A in $\mathrm{RO}(T(\mathcal{F}))$.

Notation

Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system. For all $\alpha < \lambda$, we define $i_{\alpha\lambda}$ as

$$\begin{array}{rccl} i_{\alpha\lambda} : & \mathbb{B}_\alpha & \to & C(\mathcal{F}) \\ & b & \mapsto & \langle \pi_{\beta,\alpha}(b) : \ \beta < \alpha \rangle^\frown \langle i_{\alpha\beta}(b) : \ \alpha \leq \beta < \lambda \rangle \end{array}$$

and $\pi_{\alpha\lambda}$

$$\begin{array}{rccl} \pi_{\alpha\lambda} : & T(\mathcal{F}) & \to & \mathbb{B}_\alpha \\ & f & \mapsto & f(\alpha) \end{array}$$

When it is clear from the context, we will denote $i_{\alpha\lambda}$ by i_α and $\pi_{\alpha\lambda}$ by π_α.

Fact

We may observe that:

1. *Every thread in $T(\mathcal{F})$ is completely determined by a cofinal subset of its domain. Moreover every thread in $C(\mathcal{F})$ is entirely determined by the restriction to its support.*
2. $i_{\alpha\lambda}$ *can naturally be seen as a regular embedding of* $\mathbb{B}_\alpha$ *in any of* $\mathrm{RO}(C(\mathcal{F}))$, $\mathrm{RO}(T(\mathcal{F}))$, $\mathrm{RO}(RCS(\mathcal{F}))$. *Moreover in all three cases* $\pi_{\alpha\lambda} = \pi_{i_{\alpha,\lambda}} \restriction P$ *where* $P = C(\mathcal{F}), T(\mathcal{F}), RCS(\mathcal{F})$.
3. $C(\mathcal{F})$ *inherits the structure of a boolean algebra with boolean operations defined as follows:*
    - $f \wedge g$ *is the unique thread* h *whose support* β *is the max of the support of* f *and* g *and is such that* $h(\beta) = f(\beta) \wedge g(\beta)$,
    - $\neg f$ *is the unique thread* h *whose support* β *is the support of* f *such that* $h(\beta) = \neg f(\beta)$.

Remark

WATCH OUT!!! In general $C(\mathcal{F})$ is only a subalgebra of $RO(T(\mathcal{F}))$ and not a complete subalgebra. Taking different limits yields unrelated forcings.

Fact

1. *If $g \in T(\mathcal{F})$ and $h \in C(\mathcal{F})$, we can check that $g \wedge h$ defined as the thread where eventually all coordinates α are the pointwise meet of $g(\alpha)$ and $h(\alpha)$ is the infimum of g and h in $T(\mathcal{F})$.*
2. *There can be nonetheless two distinct incompatible threads $f, g \in T(\mathcal{F})$ such that $f(\alpha) \wedge g(\alpha) > 0_{\mathbb{B}_\alpha}$ for all $\alpha < \lambda$. Thus in general the pointwise meet of two threads is not even a thread.*

Lemma

Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system and $A \subseteq T(\mathcal{F})$ be an antichain such that $\pi_{\alpha\lambda}[A]$ is an antichain for some $\alpha < \lambda$. Then $\tilde{\bigvee} A$ is the supremum of the elements of A in $\mathrm{RO}(T(\mathcal{F}))$.

Theorem (Baumgartner)

Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system such that $\mathbb{B}_\alpha$ is $< \lambda$-cc for all α and $S = \{\alpha : \mathbb{B}_\alpha \cong \mathrm{RO}(C(\mathcal{F} \restriction \alpha))\}$ is stationary. Then $C(\mathcal{F})$ is $< \lambda$-cc.

Lemma

Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be an iteration system such that $C(\mathcal{F})$ is $< \lambda$-cc. Then $T(\mathcal{F}) = C(\mathcal{F})$ is a complete boolean algebra.