Measure Recognition Problems

Piotr Borodulin–Nadzieja

Winterschool 2010, Hejnice

joint work with Mirna Džamonja

Preliminaries

Basic remarks

- we will consider finitely–additive measures on Boolean algebras;
- we will say that $(\mathfrak{A}, \mu)$ is (metrically Boolean) isomorphic to $(\mathfrak{B}, \nu)$ if there is an isomorphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ such that

$$\nu(\varphi(a)) = \mu(a)$$

for every $a \in \mathfrak{A}$;

Preliminaries

Basic remarks

- we will consider finitely–additive measures on Boolean algebras;
- we will say that $(\mathfrak{A}, \mu)$ is (metrically Boolean) isomorphic to $(\mathfrak{B}, \nu)$ if there is an isomorphism $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ such that

$$\nu(\varphi(a)) = \mu(a)$$

for every $a \in \mathfrak{A}$;

Preliminaries

Basic remarks

- we will consider finitely–additive measures on Boolean algebras;
- we will say that $(\mathfrak{A}, \mu)$ is (metrically Boolean) isomorphic to $(\mathfrak{B}, \nu)$ if there is an isomorphism $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ such that

$$\nu(\varphi(a)) = \mu(a)$$

for every $a \in \mathfrak{A}$;

Small measures

Definition

A measure μ on $\mathfrak{A}$ is *separable* if there is a countable family $\mathcal{D} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(a \triangle d) \colon d \in D\} = 0$$

for every $a \in \mathfrak{A}$.

Definition

A measure μ on $\mathfrak{A}$ is *uniformly regular* if there is a countable family $\mathcal{D} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(a \setminus d) \colon d \in D,\ d \leq a\} = 0$$

for every $a \in \mathfrak{A}$.

Small measures

Definition

A measure μ on $\mathfrak{A}$ is *separable* if there is a countable family $\mathcal{D} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(a \triangle d) \colon d \in D\} = 0$$

for every $a \in \mathfrak{A}$.

Definition

A measure μ on $\mathfrak{A}$ is *uniformly regular* if there is a countable family $\mathcal{D} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(a \setminus d) \colon d \in D,\ d \leq a\} = 0$$

for every $a \in \mathfrak{A}$.

Small measures

Definition

A measure μ on $\mathfrak{A}$ is *separable* if there is a countable family $\mathcal{D} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(a \triangle d) \colon d \in D\} = 0$$

for every $a \in \mathfrak{A}$.

Definition

A measure μ on $\mathfrak{A}$ is *uniformly regular* if there is a countable family $\mathcal{D} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(a \setminus d) \colon d \in D,\ d \leq a\} = 0$$

for every $a \in \mathfrak{A}$.

(a consequence of) Maharam's theorem

Theorem (Dorothy Maharam, 1942)

If a σ–additive measure μ on $\mathfrak{A}$ is non–atomic and separable, then $(\mu, \mathfrak{A})$ is isomorphic to $(\lambda, \mathfrak{B})$, where λ is the Lebesgue measure on the Random algebra $\mathfrak{B}$.

Problem

What about a classification of finitely–additive measures?

(a consequence of) Maharam's theorem

Theorem (Dorothy Maharam, 1942)

If a σ–additive measure μ on $\mathfrak{A}$ is non–atomic and separable, then $(\mu, \mathfrak{A})$ is isomorphic to $(\lambda, \mathfrak{B})$, where λ is the Lebesgue measure on the Random algebra $\mathfrak{B}$.

Problem

What about a classification of finitely–additive measures?

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\longleftarrow$.

Measure Recognition Problems

MRP(ϕ)

How to characterize Boolean algebras supporting a (strictly positive) measure with a property ϕ?

- MRP($\emptyset$) Kelley's theorem;
- MRP(σ–additive) Maharam's problem;
- MRP(non–atomic) Džamonja, Plebanek (2006);
- MRP(separable) ??;
- MRP(uniformly regular) ?? $\leftarrow$.

MRP(uniformly regular)

Remarks.

- assume that μ is strictly positive non–atomic uniformly regular measure on $\mathfrak{A}$;
- there is a dense countable family $\mathcal{D}$ in $\mathfrak{A}$;
- we can assume that $\mathcal{D}$ is a subalgebra of $\mathfrak{A}$ (isomorphic to the Cantor algebra);
- thus, *Cantor* $\subseteq \mathfrak{A} \subseteq$ *Cohen*;
- more precisely, if we define the Jordan algebra for μ as

$$\mathcal{J}_\mu = \{A \in \mathit{Cohen} \colon \mu_*(A) = \mu^*(A)\},$$

then $\mathfrak{A}$ is a subalgebra of $\mathcal{J}_\mu$.

MRP(uniformly regular)

Remarks.

- assume that μ is strictly positive non–atomic uniformly regular measure on $\mathfrak{A}$;
- there is a dense countable family $\mathcal{D}$ in $\mathfrak{A}$;
- we can assume that $\mathcal{D}$ is a subalgebra of $\mathfrak{A}$ (isomorphic to the Cantor algebra);
- thus, $\mathit{Cantor} \subseteq \mathfrak{A} \subseteq \mathit{Cohen}$;
- more precisely, if we define the Jordan algebra for μ as

$$\mathcal{J}_\mu = \{A \in \mathit{Cohen} \colon \mu_*(A) = \mu^*(A)\},$$

then $\mathfrak{A}$ is a subalgebra of $\mathcal{J}_\mu$.

MRP(uniformly regular)

Remarks.

- assume that μ is strictly positive non–atomic uniformly regular measure on $\mathfrak{A}$;
- there is a dense countable family $\mathcal{D}$ in $\mathfrak{A}$;
- we can assume that $\mathcal{D}$ is a subalgebra of $\mathfrak{A}$ (isomorphic to the Cantor algebra);
- thus, *Cantor* $\subseteq \mathfrak{A} \subseteq$ *Cohen*;
- more precisely, if we define the Jordan algebra for μ as

$$\mathcal{J}_\mu = \{A \in \mathit{Cohen}\colon \mu_*(A) = \mu^*(A)\},$$

then $\mathfrak{A}$ is a subalgebra of $\mathcal{J}_\mu$.

MRP(uniformly regular)

Remarks.

- assume that μ is strictly positive non–atomic uniformly regular measure on $\mathfrak{A}$;
- there is a dense countable family $\mathcal{D}$ in $\mathfrak{A}$;
- we can assume that $\mathcal{D}$ is a subalgebra of $\mathfrak{A}$ (isomorphic to the Cantor algebra);
- thus, *Cantor* $\subseteq \mathfrak{A} \subseteq$ *Cohen*;
- more precisely, if we define the Jordan algebra for μ as

$$\mathcal{J}_\mu = \{A \in \mathit{Cohen}\colon \mu_*(A) = \mu^*(A)\},$$

then $\mathfrak{A}$ is a subalgebra of $\mathcal{J}_\mu$.

MRP(uniformly regular)

Remarks.

- assume that μ is strictly positive non–atomic uniformly regular measure on $\mathfrak{A}$;
- there is a dense countable family $\mathcal{D}$ in $\mathfrak{A}$;
- we can assume that $\mathcal{D}$ is a subalgebra of $\mathfrak{A}$ (isomorphic to the Cantor algebra);
- thus, $\mathit{Cantor} \subseteq \mathfrak{A} \subseteq \mathit{Cohen}$;
- more precisely, if we define the Jordan algebra for μ as

$$\mathcal{J}_\mu = \{A \in \mathit{Cohen} \colon \mu_*(A) = \mu^*(A)\},$$

then $\mathfrak{A}$ is a subalgebra of $\mathcal{J}_\mu$.

MRP(uniformly regular)

Theorem

If a Boolean algebra supports a non–atomic uniformly regular measure, then is isomorphic to a subalgebra of (some) Jordan algebra containing the Cantor algebra.

Theorem

If μ, λ are strictly positive non–atomic measures on the Cantor algebra, then $(\mathcal{J}_\mu, \mu)$ is isomorphic to $(\mathcal{J}_\lambda, \lambda)$.

MRP(uniformly regular)

Theorem

If a Boolean algebra supports a non–atomic uniformly regular measure, then is isomorphic to a subalgebra of (some) Jordan algebra containing the Cantor algebra.

Theorem

If μ, λ are strictly positive non–atomic measures on the Cantor algebra, then $(\mathcal{J}_\mu, \mu)$ is isomorphic to $(\mathcal{J}_\lambda, \lambda)$.

Measure Recognition Problems...

MRP*

How to characterize Boolean algebras which carry **only** measures with property ϕ?

- MRP*(separable);
- MRP*(uniformly regular).

Measure Recognition Problems. . .

MRP*

How to characterize Boolean algebras which carry **only** measures with property ϕ?

- MRP*(separable);
- MRP*(uniformly regular).

Measure Recognition Problems. . .

MRP*

How to characterize Boolean algebras which carry **only** measures with property ϕ?

- MRP*(separable);
- MRP*(uniformly regular).

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable. This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Properties:

- all Boolean algebras carry a separable measure;
- If a Boolean algebra is big (i.e. it contains an ω_1 independent sequence), then it carries a non–separable measure;
- (Fremlin) under MA and non CH small Boolean algebras carry only separable measures;
- under CH (and other axioms) there is a lot of examples of small Boolean algebras with non–separable measures;
- assume μ is a strictly positive measure on $\mathfrak{A}$ and ν is a non–separable measure on $\mathfrak{A}$. Then, $\mu + \nu$ is a strictly positive non–separable measure on $\mathfrak{A}$;
- on the algebra of clopen subsets of 2^{ω_1} all strictly positive measures are non–separable.This algebra does not carry a uniformly regular measure.

MRP versus MRP*

Theorem

All Boolean algebras without a non–separable measure carry a uniformly regular measure.

Remark

Under CH there is a small Boolean algebra without a uniformly regular measure. (Talagrand's example of a *strange* Grothendieck space.)

MRP versus MRP*

Theorem

All Boolean algebras without a non–separable measure carry a uniformly regular measure.

Remark

Under CH there is a small Boolean algebra without a uniformly regular measure. (Talagrand's example of a *strange* Grothendieck space.)

MRP versus MRP*

Theorem

There is a Boolean algebra supporting a measure which does not support neither uniformly regular measure nor a non–separable one.

Proof: Bell's example of a separable compact G_δ–scattered space without a countable π–base works.

Theorem

Every minimally generated Boolean algebra supporting a measure supports a uniformly regular one.

MRP versus MRP*

Theorem

There is a Boolean algebra supporting a measure which does not support neither uniformly regular measure nor a non–separable one.

Proof: Bell's example of a separable compact G_δ–scattered space without a countable π–base works.

Theorem

Every minimally generated Boolean algebra supporting a measure supports a uniformly regular one.

MRP versus MRP*

Theorem

There is a Boolean algebra supporting a measure which does not support neither uniformly regular measure nor a non–separable one.

Proof: Bell's example of a separable compact G_δ–scattered space without a countable π–base works.

Theorem

Every minimally generated Boolean algebra supporting a measure supports a uniformly regular one.

The end

Thank you for your attention!

This research was supported by the ESF Research Networking Programme **INFTY**.

Slides and a preprint concerning the subject will be available on

`http://www.math.uni.wroc.pl/~pborod`