Introduction to Transfinite Chomp

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Before···


Nestlé
KitKat
濃い抹茶
MATCHA
宇治抹茶
10枚入
写真はイメージです
1枚当たり
63
kcal


Nestlé
KitKat
香ばしきなこに、
ほんのり桜香る。
桜きなこ
Sakura & Roasted Soybean powder
mini 10枚
写真はイメージです

$m \times n$-chomp

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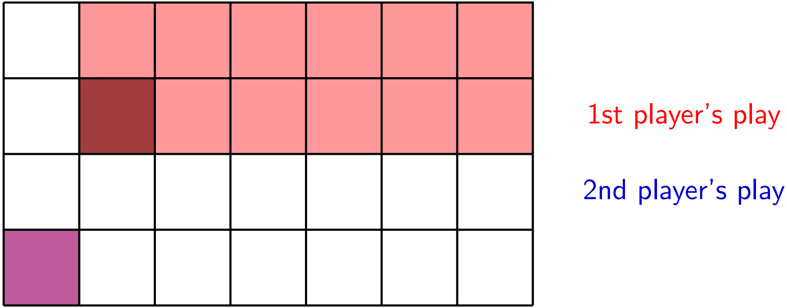

The win condition is to force the play to eat the poisoned cell $((0,0))$.

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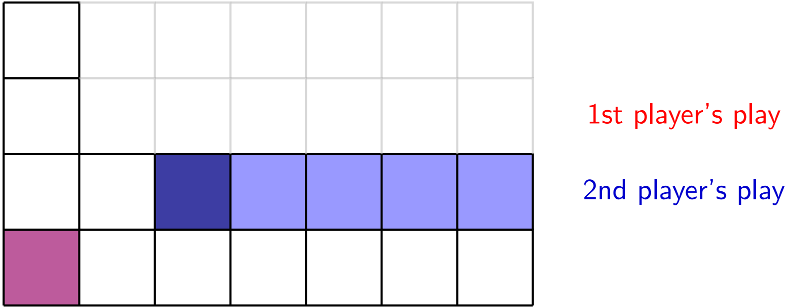

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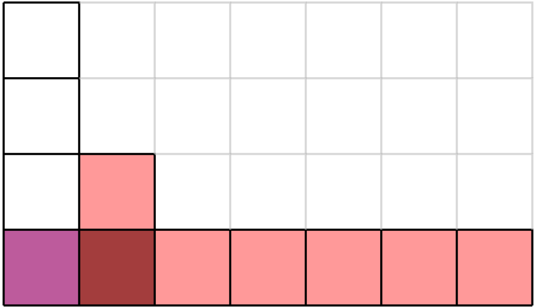

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2nd player's play

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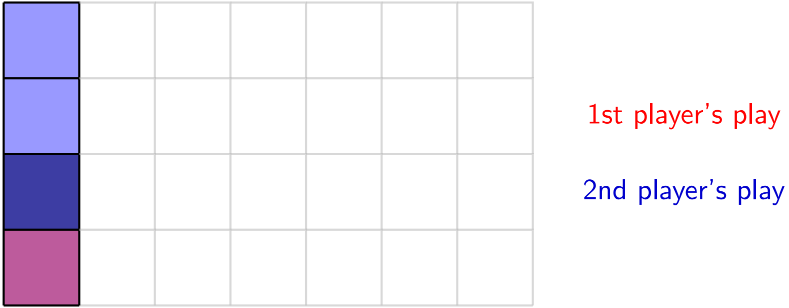

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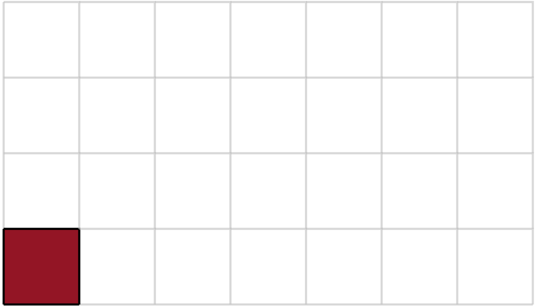

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Chomp is determined. In general, every finite two-player zero-sum game with perfect information is determined.

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- For which m, n is it a 1st player win?
- And what is an explicit winning strategy?

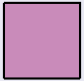
The strategy-stealing argument

Theorem

$m \times n$-chomp is a 1st player win $\iff (m, n) \neq (1, 1)$.

Proof.

$\Longrightarrow$: Prove the contrapositive. It suffices to show that 1×1-chomp is a 2nd player win.



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This is clearly a 2nd player win.

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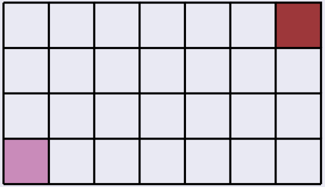
Proof (continued).

$\Longleftarrow$: By contradiction. Assume $(m, n) \neq (1, 1)$ and that the 1st player does not have a winning strategy. Then $m \times n$-chomp must be a 2nd player win.



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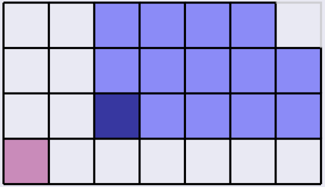

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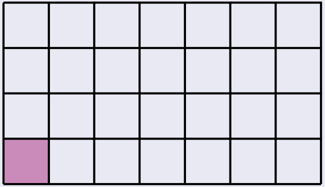

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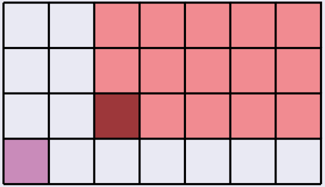


This contradicts that it was a 2nd player win. □

On explicit winning strategies

Theorem

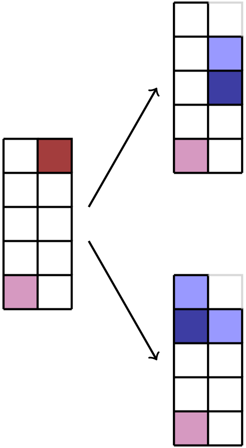
$m \times n$-chomp is a 1st player win $\iff (m,n) \neq (1,1)$.

When $(m,n) \neq (1,1)$, an explicit 1st player winning strategy for $m \times n$-chomp is generally unknown.
However, we can construct explicit winning strategies for paticular cases.

Example: $2 \times n$-chomp

Key idea:

- The 1st player can always leave a *staircase-like board state*.



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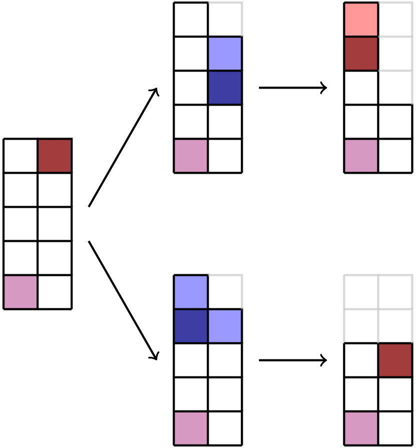

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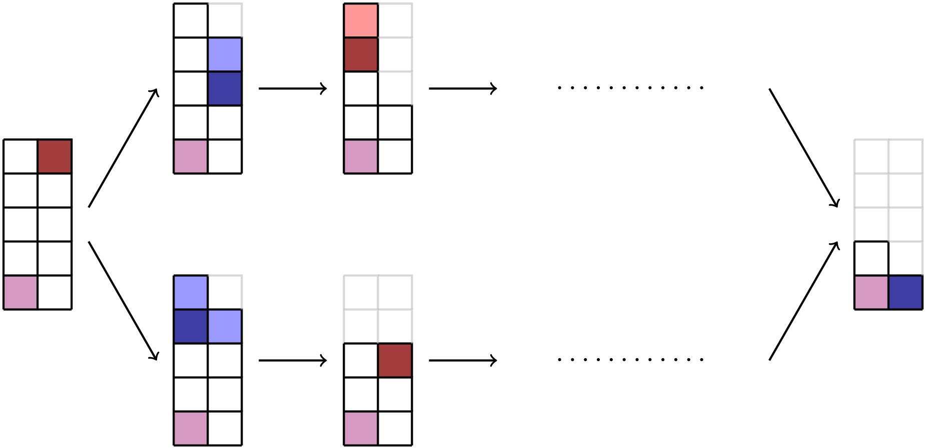
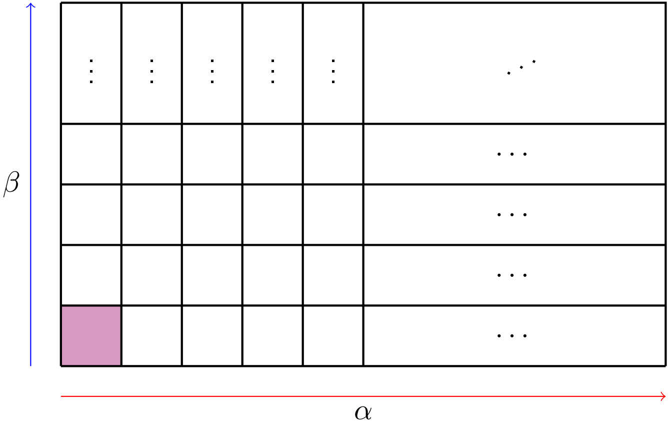


$\alpha \times \beta$-chomp

For ordinals $\alpha, \beta \geq 1$, the $\alpha \times \beta$-chomp is chomp played from the following board:



Finiteness of Transfinite Chomp

Proposition

Transfinite chomp always ends after finitely many moves. In particular, either a 1st player or a 2nd player has a winning strategy.

Of course, constructing explicit winning strategies is very very hard.

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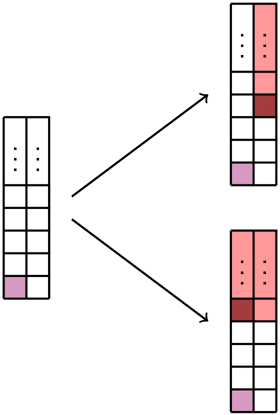
$2 \times \omega$-chomp is a 2nd player win. The strategy is as follows:



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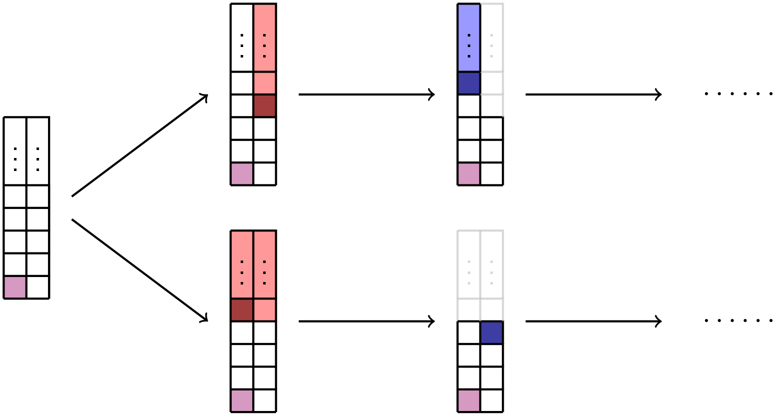

Class function f

Theorem(S. Huddleston, J. Shurman [1], Michal R. Przybylek [2] (partially), T. Goto simplified.)

For every ordinal $\alpha \geq 1$ there exists a unique $\beta \geq 1$ such that $\alpha \times \beta$-chomp is a 2nd player win.

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This defines a class function f from ordinals to ordinals by:

$$f(\alpha) = \beta \iff \alpha \times \beta \text{ is a 2nd player win.}$$

It is easy to show that $f(\alpha) = \beta \iff f(\beta) = \alpha$.

Local behavior of f

The following are known:

$$
\begin{aligned}
f(1) &= 1 \\
f(2) &= \omega \\
f(3) &= \omega^{\omega} \quad \text{(S. Huddleston, J. Shurman[1])} \\
&\vdots \\
f(\omega) &= 2 \\
&\vdots \\
f(\omega^{\omega}) &= 3 \\
&\vdots
\end{aligned}
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Global behavior of f

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Let $\mathrm{Suc}_{\omega_1} = \{\alpha < \omega_1 \mid \alpha \text{ is a successor ordinal}\}$ and $\mathrm{Lim}_{\omega_1} = \{\alpha < \omega_1 \mid \alpha \text{ is a limit ordinal}\}$. Then exactly one of $f^{-1}(\mathrm{Suc}_{\omega_1})$ or $f^{-1}(\mathrm{Lim}_{\omega_1})$ contains a club subset of ω_1, and

$$f^{-1}(\mathrm{Suc}_{\omega_1}) \text{ contains a club} \iff f(\omega_1) \text{ is a successor ordinal.}$$

Reflection

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That is, the property "$f(\alpha)$ is successor" reflects to ω_1.

Future works

- I would like to determine the value of $f(\omega_1)$.
  - At least, I would like to know whether $f(\omega_1)$ is a successor or a limit ordinal.
  - By our second proposition, if we can compute $f(\alpha)$ for many $\alpha < \omega_1$, we can answer this.

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Summary

PLEASE HELP ME!!!

Thank you for your attention!

References I

[1] Scott Huddleston and Jerry Shurman.
Transfinite chomp.
In *More Games of No Chance, Proc. MSRI Workshop on Combinatorial Games*, pp. 183–212, 2000.

[2] Michal R. Przybylek.
Infinite chocolate.
2019.