Stuctural IEPRs and Weak Choice Principles

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Winter School, Hejnice
2nd of February 2026

Contains joint work with Thilo Weinert and Jonathan Schilhan

Structural partition relations

Notation: Copies of B in A

For A and B structures in some language, write $[A]^B$ for the set of subsets of A which are isomorphic to B when thought of as (induced) substructures.

Definition: Partition relation symbol

For A, B, C structures in some language and χ a set,

$$A \to (B)^C_\chi$$

is the statement that for any $F : [A]^C \to \chi$, thought of as a *colouring* of the copies of C in A, there is some $H \in [A]^B$ which is *homogeneous* or *monochromatic* for F, in the sense that $\left|F \text{ " } [H]^C\right| = 1$.

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Notation: two-colourings

When $\chi = 2$ it is usually omitted from the notation, so

$$A \to (B)^C \text{ means the same thing as } A \to (B)^C_2.$$

Non-structural example: Ramsey's theorem

The infinitary version of Ramsey's theorem is the statement that for any $n, m \in \omega$,

$$\omega \to (\omega)^n_m.$$

Infinite-exponent partition relations and the Axiom of Choice

- [ER52] Under AC,

$$(\forall\kappa)\ \kappa \not\to (\omega)^\omega.$$

- [Ma70] In Solovay's model of ZF + DC,

$$\omega \to (\omega)^\omega.$$

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Proposition 1 (G., [Ga25])

Let A, B be structures in some language such that $[B]^B \neq \{B\}$, i.e. such that B contains proper subcopies of itself. Then under AC,

$$A \not\to (B)^B.$$

As such, our base theory throughout this talk is ZF.

Proof

Well-order $[A]^B$ as $\langle B_\alpha : \alpha < \kappa \rangle$, some κ. Then define a colouring $F : [A]^B \to 2$ with no homogeneous set inductively: at stage α, if the value of $F(B_\alpha)$ has not yet been determined, try to find a sequence

$$B_\alpha = B_{\alpha_0} \supsetneq B_{\alpha_1} \supsetneq B_{\alpha_2} \supsetneq \dots$$

such that no $F(B_{\alpha_n})$ has been determined; then set $F(B_{\alpha_n}) = 0$ if n is even, $F(B_{\alpha_n}) = 1$ if n is odd. Say that the B_{α_n} have been *coloured by alternation*. If it is not possible to find such a sequence, set $F(B_\alpha) = 0$, and note that this can only happen if some element of $[B_\alpha]^B$ has already been coloured by alternation by stage α. By construction, any B_α which has been coloured by alternation cannot be homogeneous for F, so no element of $[A]^B$ is homogeneous for F. □

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IEPRs on Linear Orders

- Main focus so far: linear orders.
- Recently fully classified the relation

$$\langle {}^{\alpha}2, <_{\mathsf{lex}} \rangle \to (\tau)^{\tau}$$

for τ countable.

Failures of weakenings of Choice

- For some τ,

$$(\forall \alpha \in \mathrm{Ord})\ \langle {}^{\alpha}2, <_{\mathsf{lex}} \rangle \not\to (\tau)^{\tau},$$

but consistently there exists *some* $\langle L, < \rangle$ with

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- These $\langle L, < \rangle$ give rise to local failures not just of AC but of even weaker choice principles.

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Kinna-Wagner 1 and the Ordering Principle

Definition: The Kinna-Wagner Selection Principle (KWP_1)

KWP_1 is the following statement:

$$\forall X\ \exists\alpha \in \mathrm{Ord} \text{ such that } X \text{ injects into } \mathcal{P}(\alpha).$$

This is a choice principle which is strictly weaker than AC (KWP_0).
It strictly implies the following principle:

Definition: Ordering Principle (O)

The *Ordering Principle* O is the statement that every set can be linearly ordered.

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Definition: scattered order

A linear order is said to be *scattered* if it has no subset which is densely ordered.

Proposition (G., Schilhan, Weinert)

1. Let τ be an order type such that $\tau + \tau \leq \tau$. Then for all $\alpha \in \mathrm{Ord}$,
$$\langle {}^{\alpha}2, <_{\mathrm{lex}} \rangle \not\to (\tau)^{\tau}.$$
2. Let τ be a countable scattered order type such that $\omega\omega^* \leq \tau$ or $\omega^*\omega \leq \tau$. Then for all $\alpha \in \mathrm{Ord}$,
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For τ countable, $\tau + \tau \leq \tau$ iff τ is non-scattered.

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Proposition (G., Schilhan)

It is consistent that there exists $\langle L, <\rangle$ such that for any countable τ, $\langle L, <\rangle \to (\tau)^\tau$.

The proof finds such an $\langle L, <\rangle$ in a symmetric extension; the elements of L are sets of ω_1-Cohen reals.

Question

Is it consistent that there exists such an $\langle L, <\rangle$ such that the elements of L are just sets of ordinals, as opposed to sets of sets of ordinals, or must any such $\langle L, <\rangle$ witness a failure of KWP_1?

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Results (G., Schilhan)

1. Let τ be a countable non-scattered order type and suppose $\langle L, <\rangle$ is such that
$$\langle L, <\rangle \to (\tau)^{\tau}.$$
Then the set L does not inject into the power set of an ordinal.
2. Let τ be a well-orderable scattered order type with $\omega\omega^* \leq \tau$ or $\omega^*\omega \leq \tau$, and suppose $\langle L, <\rangle$ is such that
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3. Let R denote the random (Rado) graph, and suppose G is a graph such that
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