Almost disjoint families under automorphisms of $\wp(\mathbb{N})/Fin$ and ℓ_∞/c_0

Małgorzata Rojek
(based on joint work with Piotr Koszmider)

Insitute of Mathematics, Polish Academy of Sciences
Faculty of Mathematics, Informatics and Mechanics, University of Warsaw

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Basic notions – almost disjoint families

Definition

A family $\mathcal{A}$ of infinite subsets is said to be *almost disjoint (AD)* if $A \cap B$ is finite for any $A \neq B$ from $\mathcal{A}$.

- Throughout this talk, we will only consider infinite (usually even uncountable) AD families of subsets of $\mathbb{N}$.
- We will "ignore" finite sets. If elements of two AD families differ on some finite sets, we would not distinguish them.

- $Fin = \{a \subseteq \mathbb{N} : |a| < \omega\}$ is the ideal of finite subsets of $\mathbb{N}$
- $\pi : \wp(\mathbb{N}) \to \wp(\mathbb{N})/Fin$ is the quotient map

Definition

We would denote by $Fincofin(\kappa)$ the Boolean algebra of finite and cofinite subsets of κ (where κ is a cardinal number).

Remark

If $\mathcal{A}$ is an AD family, then $\pi[\mathcal{A}]$ is a disjoint family; and the Boolean subalgebra of $\wp(\mathbb{N})/Fin$ generated by $\pi[\mathcal{A}]$ is isomorphic to $Fincofin(|\mathcal{A}|)$. And vice versa, any subalgebra of $\wp(\mathbb{N})/Fin$ isomorphic to $Fincofin(\kappa)$ is generated by a (unique) disjoint family $\mathcal{A}^*$, which can be lifted to some AD family in $\mathbb{N}$.

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Question: Are all AD families of a given cardinality "the same"?

Are all AD families the same?

Definition

Let $\mathbb{A}$ and $\mathbb{B}$ be two structures. We would say that $\mathbb{A}$ is $\mathbb{B}$-*automorphic* if for any substructures $\mathbb{B}_1, \mathbb{B}_2$ isomorphic to $\mathbb{B}$ of the same codimension (i.e. $\mathrm{dens}(\mathbb{A}/\mathbb{B}_1) = \mathrm{dens}(\mathbb{A}/\mathbb{B}_2)$) any isomorphism between $\mathbb{B}_1$ and $\mathbb{B}_2$ extends to an automorphism of $\mathbb{A}$.

Question

Is $\wp(\mathbb{N})/\mathit{Fin}$ a *Fincofin*(ω_1)-automorphic algebra?

Question' (equivalent)

Is it true that for any two AD families $\mathcal{A}, \mathcal{B}$ of cardinality ω_1 there exists a Boolean automorphism h of $\wp(\mathbb{N})/\mathit{Fin}$ such that $h \circ \pi|_{\mathcal{A}} = \pi|_{\mathcal{B}}$?

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Question

Is $\wp(\mathbb{N})/\mathit{Fin}$ a $\mathit{Fincofin}(\omega_1)$-automorphic algebra?

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Is it true that for any two AD families $\mathcal{A}, \mathcal{B}$ of cardinality ω_1 there exists a Boolean automorphism h of $\wp(\mathbb{N})/\mathit{Fin}$ such that $h \circ \pi|_{\mathcal{A}} = \pi|_{\mathcal{B}}$?

Answer: no.

Example of two families

Given two families $\mathcal{B}, \mathcal{C}$, we say that a set $X \subseteq \mathbb{N}$ separates $\mathcal{B}$ and $\mathcal{C}$ if $B \subseteq^* X$ for every $B \in \mathcal{B}$ and $A \cap X =^* \emptyset$ for every $A \in \mathcal{C}$.

Luzin family

There is an AD family of cardinality ω_1 such that no two uncountable subfamilies can be separated.

"Cantor family"

Branches of the tree $2^{<\omega}$ constitute an AD family on a countable set. Every uncountable subfamily of this family has many separations.

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Observation: one of these almost disjoint families is contained in some bigger algebra, the other one is not.
Notion: partition algebras?

Setting – Banach spaces

We define:

$$\ell_\infty(\Gamma) = \{(x_\alpha)_{\alpha\in\Gamma} \in \mathbb{R}^\Gamma : x_\alpha \text{ is bounded}\}$$

$$c_0(\Gamma) = \{(x_\alpha)_{\alpha\in\Gamma} \in \mathbb{R}^\Gamma : \forall_{\varepsilon>0}\, \exists_{F\in[\Gamma]^{<\omega}}\, |x_\alpha| < \varepsilon \text{ for } \alpha \in \Gamma \setminus F\}$$

both considered with the supremum norm. In particular:

- $\ell_\infty = \ell_\infty(\mathbb{N})$ is the space of bounded sequences
- $c_0 = c_0(\mathbb{N})$ is the (sub)space of sequences convergent to 0
- ℓ_∞/c_0 is the quotient space with the quotient norm and $\bar{\pi} : \ell_\infty \to \ell_\infty/c_0$ is the quotient map
- for $f, g \in \ell_\infty$ we would also write $f =^* g$ iff $f - g \in c_0$ or, equivalently, $\bar{\pi}(f) = \bar{\pi}(g)$

Side remark

$$\ell_\infty \equiv C(\beta\mathbb{N}) \equiv C(S(\wp(\mathbb{N}))$$

$$\ell_\infty/c_0 \equiv C(\mathbb{N}^*) \equiv C(S(\wp(\mathbb{N})/Fin))$$

Remark

We have natural correspondences between sets (elements of algebra) and characteristic functions (vectors in Banach spaces):

- $\wp(\mathbb{N}) \ni X \mapsto 1_X \in \ell_\infty$
- $\wp(\mathbb{N})/Fin \ni \pi(X) \mapsto 1_{\pi(X)} = \bar{\pi}(1_X) \in \ell_\infty/c_0$

Definition

Let $\mathcal{A}$ be an AD family. We define the following space:

$$Y_{\mathcal{A}} = \overline{\text{span}}\{1_{\pi(A)} : A \in \mathcal{A}\} \subseteq \ell_\infty/c_0$$

Remark

$Y_{\mathcal{A}}$ is isometrically isomorphic to $c_0(\omega_1)$

Remark

Isometric isomorphisms T of ℓ_∞/c_0 correspond to Boolean automorphisms h of $\wp(\mathbb{N})/Fin$:

- if h is a Boolean automorphism then the formula $T_h(1_B) = 1_{h(B)}$ defines an isometric isomorphism of ℓ_∞/c_0
- if T is an isometric automorphism of ℓ_∞/c_0, then $h_T(B) = \mathrm{supp}(T(1_B))$ defines an automorphism of $\wp(\mathbb{N})/Fin$

Corollary

There exist two AD families $\mathcal{A}, \mathcal{B}$ such that there is no isometric automorphism T of ℓ_∞/c_0 with the property that $T[Y_\mathcal{A}] = Y_\mathcal{B}$. So ℓ_∞/c_0 is not isometrically $c_0(\omega_1)$-automorphic.

Banach space automorphisms

Question 1

Is it true for any two AD families $\mathcal{A}, \mathcal{B}$ of cardinality ω_1 there exists an automorphism T of ℓ_∞/c_0 (not necessarily isometric) that sends $\{1_{\pi(A)} : A \in \mathcal{A}\}$ onto $\{1_{\pi(B)} : B \in \mathcal{B}\}$?

Question 2

Is ℓ_∞/c_0 a $c_0(\omega_1)$-automorphic?

(the property from question 2 is stronger)

A useful space

We denote:

$$Cc(\kappa) := \{A \subseteq \kappa : A \text{ is countable or cocountable}\}$$

$$\ell_\infty^c(\kappa) := \{f \in \ell_\infty(\kappa) : f \text{ is countably supported}\}$$

Remark

- characteristic functions 1_A for countable $A \subseteq \kappa$ are linearly dense in $\ell_\infty^c(\kappa)$. As a consequence, any Boolean embedding of $Cc(\kappa)$ into $\mathbb{A}$ induces an isometric embedding embedding of $\ell_\infty^c(\kappa)$ into $C(S(\mathbb{A}))$
- $c_0(\kappa)$ is a subspace $\ell_\infty^c(\kappa)$ (it is $\overline{\text{span}}\{1_{\{\alpha\}} : \alpha < \kappa\}$)

Theorem (Koszmider, R. 2025)

There is an isomorphic Boolean embedding $h : Cc(\omega_1) \hookrightarrow \wp(\mathbb{N})/Fin$. Consequently, there is an isometric embedding $T : \ell_\infty^c(\omega_1) \hookrightarrow \ell_\infty/c_0$.

Theorem (Koszmider, R. 2025)

If $\mathcal{A} := \{A_\xi : \xi < \kappa\}$ is a maximal almost disjoint family then there is no bounded operator $S : \ell_\infty^c(\kappa) \to \ell_\infty/c_0$ such that $S(1_{\{\xi\}}) = 1_{\pi(A_\xi)}$.

Corollary

If there exists a MAD family of cardinality ω_1, then the space ℓ_∞/c_0 is not $c_0(\omega_1)$-automorphic.

Theorem (Koszmider, R. 2025)

There is an isomorphic Boolean embedding $h : Cc(\omega_1) \hookrightarrow \wp(\mathbb{N})/Fin$. Consequently, there is an isometric embedding $T : \ell_\infty^c(\omega_1) \hookrightarrow \ell_\infty/c_0$.

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Corollary

If there exists a MAD family of cardinality ω_1, then the space ℓ_∞/c_0 is not $c_0(\omega_1)$-automorphic.

Question: could we find similar examples in ZFC?

Theorem (Koszmider, R. 2025)

Assume $\mathrm{MA}_\kappa(\sigma\text{-centered})$. Then for every two almost disjoint families $\mathcal{A}, \mathcal{B}$ of cardinality κ and any bijection $s : \mathcal{A} \to \mathcal{B}$ there exists an automorphism $T : \ell_\infty \to \ell_\infty$ such that $T[c_0] = c_0$ and $T(1_A) =^* 1_{s(A)}$.

Theorem (more general) (Koszmider, R. 2025)

Assume $\mathrm{MA}_\kappa(\sigma\text{-centered})$. Then for every two subspaces Y_1, Y_2 of ℓ_∞/c_0 isomorphic to $c_0(\kappa)$ and any isomorphism $S : Y_1 \to Y_2$ there exists an automorphism $T : \ell_\infty \to \ell_\infty$ such that $T[c_0] = c_0$ and the automorphism of ℓ_∞/c_0 induced by T is an extension of S.

Corollary

Under $\mathrm{MA}_\kappa(\sigma\text{-centered})$ the space ℓ_∞/c_0 is $c_0(\omega_1)$-automorphic.

Theorem

$\wp(\mathbb{N})/\mathit{Fin}$ is countably automorphic, i.e., it is $\mathbb{B}$-automorphic for any countable Boolean algebra $\mathbb{B}$.

But, as we have seen, there are algebras $\mathbb{B}$ of cardinality ω_1 such that $\wp(\mathbb{N})/\mathit{Fin}$ is not $\mathbb{B}$-automorphic (e.g. $\mathbb{B} = \mathit{Fincofin}(\omega_1)$).
Also:

Theorem (Avilés, Cabello Sánchez, Castillo, González, Moreno)

ℓ_∞/c_0 is separably automorphic, i.e., it is Y-automorphic for any separable Banach space Y.

Further questions

1. For what other spaces Y the space ℓ_∞/c_0 is consistently Y-automorphic?
2. Is it consistent that ℓ_∞/c_0 is Y-automorphic for any Banach space Y of density ω_1?
3. Is it provable in ZFC that ℓ_∞/c_0 is not $\mathfrak{a}$-automorphic? If yes, how many classes of AD families of size $\mathfrak{a}$ are there (if the classes consist of families which can be mapped one onto another by some automorphism of ℓ_∞/c_0)?
4. What kind of properties of AD families can be used to distinguish them on the level of Banach spaces?

Thank you for your attention!

Some references:

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