Baire Category Invariants and the Structure of Non-Separable Banach Spaces

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Winter School in Abstract Analysis: Section Set Theory and Topology

Introduction

This presentation will survey Todorčević's article "Biorthogonal systems and quotient spaces via Baire category methods".

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Biorthogonal systems and quotient spaces via baire category methods

Stevo Todorcevic

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Abstract. We show that every Banach space X of density smaller that the Baire category number $\mathfrak{m}\mathfrak{m}$ admits a quotient with a long Schauder basis that can be taken of length ω_1 if X is not separable. So, assuming that the Baire category number $\mathfrak{m}\mathfrak{m}$ does not take its minimal possible value, a Banach space X is separable if and only if all biorthogonal systems of X are countable.

The classical definition of a Schauder basis is as follows:

Definition 1: Schauder basis

A sequence $(x_n)_{n\in\omega}$ in a Banach space X is a **Schauder basis** if for every $x \in X$ there exists a unique sequence of scalars $(\alpha_n)_{n\in\omega}$ such that

$$x = \sum_{n\in\omega} \alpha_n x_n.$$

Motivation

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We know that the existence of this type of basis is restricted to separable spaces. Furthermore, not all separable spaces have it.

Motivation

Fortunately we can go to closed subspaces or to quotient spaces.

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Every separable Banach space X has a subspace Y such that X/Y has a Schauder basis.

But what about non-separable spaces?

(Long) Schauder bases

A possible generalization of Schauder bases to non-separable spaces is the following:

Definition 2: (Long) Schauder basis

Let X be a Banach space and Γ be an ordinal. A sequence $(x_\alpha)_{\alpha<\Gamma}$ in X is a **(long) Schauder basis** if for every $x \in X$ there exists a unique sequence $(\alpha_\beta)_{\beta<\Gamma} \subseteq \mathbb{K}$ such that

$$x = \sum_{\beta<\Gamma} \alpha_\beta x_\beta.$$

The question

Does every non-separable Banach space X have a subspace Y such that X/Y has an uncountable basis?

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Todorčević proved that this is true under PID when $d(X) < \mathfrak{m}$.

An important lemma

Lemma 1: Todorčević

For a given Banach space X, there is a bounded linear operator $H : X \longrightarrow c_0(\omega_1)$ whose range is non-separable if and only if there is a normalized sequence $(f_\gamma)_{\gamma\in\omega_1}$ of bounded linear functionals on X such that $\left|\bigcup_{\beta\in\omega_1} supp(f_\gamma)\right| \geq \omega_1$ and $\{f_\gamma(x) : \gamma \in \omega_1\} \in c_0(\omega_1)$ for all $x \in X$.

The main ideas

The core argument lies in the following theorem:

Theorem 3: Todorčević

Suppose X is a Banach space of density $< \mathfrak{m}$ admitting a bounded linear operator $H : X \longrightarrow c_0(\omega_1)$ with nonseparable range. Then X admits a quotient with a basis of length ω_1.

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The core argument lies in the following theorem:

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Along its proof, Baire category arguments are used twice. Once to prove that the following space is norm-dense in X:

$$Y = \left\{ x \in X : \sum_{\alpha < \omega_1} |f_\alpha(x)| < \infty \right\}$$

The forcing poset

...and again to find an uncountable subsequence of $S = (f_\alpha)_{\alpha\in\Gamma}$ such that there is a quotient map from X onto $\overline{\text{span}}S$.

The forcing poset we use here is the set of the $p = (D_p, \Gamma_p, \varepsilon_p)$ such that:

- D_p is a finite subset of Y;

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- D_p is a finite subset of Y;
- Γ_p is a finite subset of ω_1;
- ε_p is a rational number in $(0, 1)$;
- For every $f^* \in (\text{span}\{f_\gamma : \gamma \in \Gamma_p\})^*$ with $\|f^*\| = 1$, there is an $x \in D_p$ with $\|x\| = 1$ such that
$$|f^*(e) - e(x)| \leq \frac{\varepsilon_p}{3}\|e\|$$
for all $e \in \text{span}\{f_\gamma : \gamma \in \Gamma_p\}$.

The forcing poset

We order the poset by letting $p \leq q$ if explicitly:

- $D_p \subseteq D_q$, $\Gamma_p \subseteq \Gamma_q$ and $\varepsilon_p \geq \varepsilon_q$;

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$$(\forall x \in D_p) \quad \Gamma_{\varepsilon_p/3}(x) \cap (\Gamma_q \setminus \Gamma_p) = \emptyset.$$

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where $\Gamma_\varepsilon(x)$ is a finite set such that $\sum_{\alpha \notin \Gamma_\varepsilon(x)} |f_\alpha(x)| < \varepsilon$.

Why is it ccc?

Theorem 4: Lindenstrauss–Rosenthal

Let X be a Banach space. For every finite-dimensional space E of X^{**}, finite-dimensional subspace F of X^* and $\varepsilon > 0$ there exists a linear isomorphism $T : E \longrightarrow T(E) \subseteq X$ such that $||T|| \cdot ||T^{-1}|| \leq 1 + \varepsilon$, $x^*(Tx^{**}) = x^{**}(x^*)$ for all $x^{**} \in E$ and $x^* \in F$ and T is the identity on $E \cap X$.

The dense subsets

Let Y_0 be a dense subset of Y of size $< \mathfrak{m}$. We now define the dense subsets of our poset:

- For every $x \in \operatorname{span}_{\mathbb{Q}} Y_0$, the set $D_x = \{p \in \mathbb{P} : x \in D_p\}$ is dense;

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- For every $\gamma < \omega_1$, the set $E_\gamma = \{p \in \mathbb{P} : (\exists \alpha \in \Gamma_p)\ \alpha \geq \gamma\}$ is dense;
- For every rational $\varepsilon \in (0, 1)$, the set $F_\varepsilon = \{p \in \mathbb{P} : \varepsilon_p \leq \varepsilon\}$ is dense.

What about the operator?

Theorem 5: Todorčević

Let X be a Banach space **of density** $< \mathfrak{m}$ whose dual ball equipped with the weak* topology **is** countably determined. Then there is a bounded linear operator with non-separable range $H : X \longrightarrow c_0(\omega_1)$.

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Let X be a Banach space **of density** $< \mathfrak{m}$ whose dual ball equipped with the weak* topology **is** countably determined. Then there is a bounded linear operator with non-separable range $H : X \longrightarrow c_0(\omega_1)$.

Theorem 6: Todorčević

Let X be a Banach space whose dual ball equipped with the weak* topology **is not** countably determined. Then, **under PID** there is a bounded linear operator with non-separable range $H : X \longrightarrow c_0(\omega_1)$.

Final remarks

- Is it true that every Banach space has a quotient space with a Schauder basis of the same length as its density? Unfortunately, **no**. One example is the space $l_\infty(2^{\aleph_0})$.

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- Is it true that every Banach space has a quotient space with a Schauder basis of the same length as its density? Unfortunately, **no**. One example is the space $l_\infty(2^{\aleph_0})$.
- This question is equivalent to the one about the existence of bounded fundamental biorthogonal systems (Plichko).

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Thank you!

Obrigado!

Děkuji!