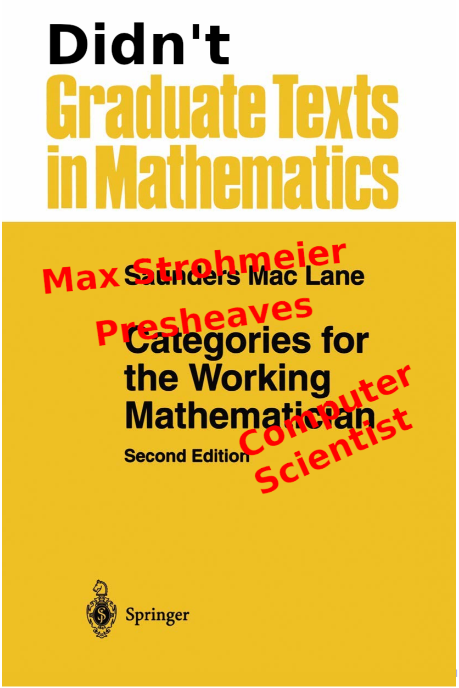
Sheaves for Ramsey Theory and Computer Science

Talk for the Winter School in Abstract Analysis

Maximilian Strohmeier

06.02.2026


Didn't
Graduate Texts in Mathematics
Max Strohmeier
Saunders Mac Lane
Presheaves
Categories for the Working Mathematician
Computer Scientist
Second Edition
Springer

Section 1

What is $C \to (B)^{A}_{r,d}$?

Definition (Ramsey property).

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$$\left.\begin{array}{l} \forall A, B \in K, r \in \omega : \\ \exists C \in K : \\ \forall c \in r^{K(A,C)} : \\ \left.\begin{array}{l} \exists b \in K(B,C), d \hookrightarrow r : \\ \forall a \in K(A,B) : \\ c(b \circ a) \in d. \end{array}\right\} \begin{array}{l} \exists b \in K(B,C) : \\ \left| c\,(b \circ -)\,[K(A,B)] \right| = d \end{array} \end{array}\right\} C \to (B)^A_{r,d}$$

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(b) *$A \in K$ has small Ramsey degree d if d is minimal such that*

$$\forall B \in K, r \in \omega : \exists C \in K : C \to (B)^A_{r,d}.$$

We say K has finite small Ramey degrees if all $A \in K$ have finite small Ramey degree.

Section 2

Presheaves

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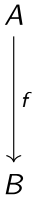

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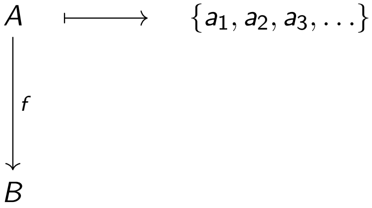

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$$\begin{array}{ccc} A & \longmapsto & \{a_1, a_2, a_3, \ldots\} \\ \downarrow{\scriptstyle f} & & \\ B & \longmapsto & \{b_1, b_2, b_3, \ldots\} \end{array}$$

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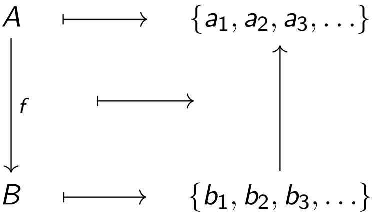

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$$\exists C \in K : C \to (B)^A_{r,d}$$
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$\pi F\{A \Rrightarrow B\}$ *is covered by d partial cones*
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Section 3

But what if I *want* confluence?

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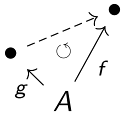

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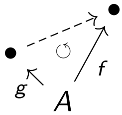
$f \leq g$:⇔ 

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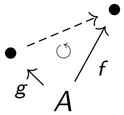

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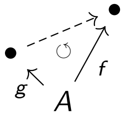
$f \leq g \;:\Leftrightarrow$ 

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Let K be a small category, $A \in K$ and $r, d \in \omega$ such that

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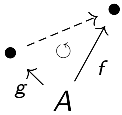

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$\Rightarrow \forall F : K^{\mathrm{op}} \to \mathbf{Set}_{\neq\emptyset}, \pi : F(A) \to r : \pi FA_*$ is covered by d partial cones in $\mathbf{Set}_{\neq\emptyset}$ with non-empty intersection everywhere.

Corollary.

Let K be a category.

Let $A \in K$ and A_* locally finite. $$\mathrm{rd}(A) = d$$ $$\Updownarrow$$ $$\forall B \in K, F : K^{\mathrm{op}} \to \mathbf{Set}_{\neq\emptyset} :$$ *$F\{A \Rrightarrow B\}$ is covered by d partial cones with non-empty intersection in $F(B)$.*	*Let $A, B \in K$, $r \in \omega$ and $\lvert K(A,A)\rvert = 1$.* $$\exists C \in K : C \to (B)^A_{r,1}$$ $$\Updownarrow$$ $$\forall F : K^{\mathrm{op}} \to \mathbf{Set}_{\neq\emptyset} : \lvert F(A)\rvert \leq r \Rightarrow \exists \varprojlim F\{A \Rrightarrow B\} \neq \emptyset.$$
Let $A \in K$ and A_* locally finite. $$A \text{ confluent} + \mathrm{rd}(A) = d$$ $$\Downarrow$$ *$\forall F : K^{\mathrm{op}} \to \mathbf{Fin}_{\neq\emptyset} : FA_*$ is covered by d partial cones with non-empty intersection everywhere* $$\Downarrow$$ *A_* is d-c.c. $+ \mathrm{rd}(A) = d$.*	*(Hadek; [Had25, Thm. 1.2])* *Let K be small and locally finite.* $$\text{Confluence} + \textit{Ramsey}$$ $$\Updownarrow$$ $$\forall F : K^{\mathrm{op}} \to \mathbf{Fin}_{\neq\emptyset} : \exists \lim F \neq \emptyset$$

Section 4

How to imagine this?

Example.

$$\text{Graphs in a class } K \supseteq \left\{ \{\bullet\} \overset{s}{\underset{t}{\rightrightarrows}} \{\bullet - \bullet\} \right\} \text{ are } r\text{-colorable}$$

$$\Updownarrow$$

$$\neg \exists C \in K : C \to (\{\bullet - \bullet\})^{\{\bullet\}}_{r,1}$$

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$$\Updownarrow$$

$$\exists F : K^{\mathrm{op}} \to \mathbf{Fin}_{\neq \emptyset} : |F(\{\bullet\})| \leq r \wedge$$
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A presheaf witnessing r-colorability and general r-coloring instructions for each graph which can be directly translated to another in the following way:

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Vice versa, if we have a class K of r-colorable finite graphs send each graph to the set of r-colorings. Each graph embedding $H \subseteq G$ we send to the function mapping r-colorings of G to their restrictions to H. □

Section 5

And how do you use it?

Theorem (Slightly generalized from [NEŠ05, Thm. 4.2(i)]).

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$$F : K^{\mathrm{op}} \to \mathbf{Fin}$$
$$D \mapsto \left\{ \chi \in \{b, c\}^{K(A,D)} \;\middle|\; \forall f \in K(A, D), g : g \circ \chi(f) \neq f \right\},$$

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Thank you for paying attention to my propaganda.

Maximilian Hadek.
Kőnig = ramsey, a compactness lemma for ramsey categories, 2025.