Modern methods of constructing the Knaster continuum

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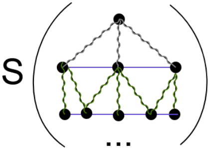
CHAPTER 0

Introduction



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Plan for today: Describe the Knaster continuum using three limit-like constructions: inverse limit, projective Fraïssé limit, spectrum of an ω-poset.

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Plan for today: Describe the Knaster continuum using three limit-like constructions: inverse limit, projective Fraïssé limit, spectrum of an ω-poset.

Main idea: Describe well-known, classical topological spaces using new tools, involving abstract language of category theory and study their properties from different perspectives.

But why?

- satysfying combinatorics, nice pictures
- results on spaces
  - a new continuum has been discovered [CKRY]
  - new characterizations of the pseudoarc [IS], [BK]
  - tools to study interesting properties such as e.g. homogeneity
- results on homeomorphisms groups

Why projective Fraïssé ?

Define **the universal Knaster continuum** K to be a Knaster continuum which continuously and openly surjects onto all other Knaster continua.

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Theorem, (S. Iyer, [I-1], 2022)

The group $Homeo(K)$ is isomorphic as a topological group to

$$U \rtimes F$$

where U is a Polish extremely amenable group and F is the free abelian group on countably many generators.

Theorem, (S. Iyer, [I-2], 2023)

The group $Homeo(K)$ contains an open subgroup with a comeager conjugacy class.

Why posets and spectra?

Recall that the **pseudo-arc** P is the hereditarily indecomposable chainable continuum.

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Theorem, (T.Bice, M.Malicki, [BM], 2024)

Homeo(P), the homeomorphism group of the pseudoarc, has a dense conjugacy class.

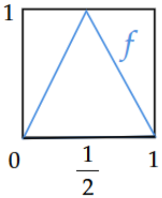
CHAPTER 1

The *continous*, inverse limit incarnation of the Knaster continuum

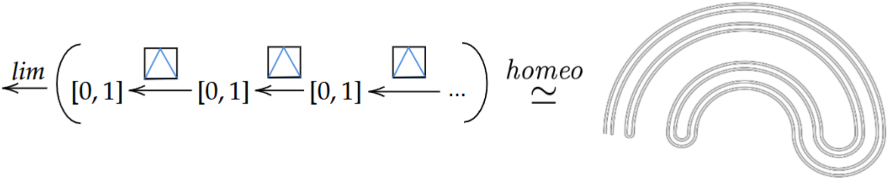
with morphisms being functions $f : I \to I$, for $I =: [0, 1]$

The Knaster continuum

Recall that **the Knaster continuum** is a space homeomorphic to the inverse limit of the sequence of arcs $\varprojlim([0,1], f_i)_{i=1}^{\infty}$, where for each i, $f_i = f$ (depicted above)



The Knaster continuum



The Knaster continuum (K_2) and its family - Knaster continua [I-1]

A (generalized) Knaster continuum is a continuum of the form

$$\varprojlim(I_n, f_n)$$

where each $I_n = [0, 1]$ and f_n is an open, continuous surjection.

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Special cases of (generalized) Knaster continua

- The double-splitting Knaster continuum K_2 (to which we will refer as to: *the Knaster continuum*)
- The universal Knaster continuum - a (generalized) Knaster continuum which continuously and openly surjects onto all other (generalized) Knaster continua

CHAPTER 2

The *discrete*, projective Fraïssé-theoretic incarnation of the Knaster continuum

with morphisms being functions $f_n : K_{n+1} \to K_n$,
for $K_n =:$ finite linear graph endowed with discrete topology

A few definitions

- For us a **graph** is a set A equipped with a symmetric, reflexive relation R. An **epimorphism** is a map between graphs which preserves the relation and is surjective on vertices and edges.

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- A **projective Fraïssé category**, $\mathcal{F}$, is a countable (up to isomorphism) category of finite graphs and morphisms s.t.
  1. each morphism in $\mathcal{F}$ is an epimorphism,
  2. $\mathcal{F}$ satisfies the joint projection property (JPP),
  3. $\mathcal{F}$ satisfies the projective amalgamation property (AP).

Projective Fraïssé limits

For a projective Fraïssé category $\mathcal{F}$, let $\mathcal{F}^\omega$ be all topological graphs formed as inverse limits of a sequence of structures in $\mathcal{F}$ via morphisms in $\mathcal{F}$.

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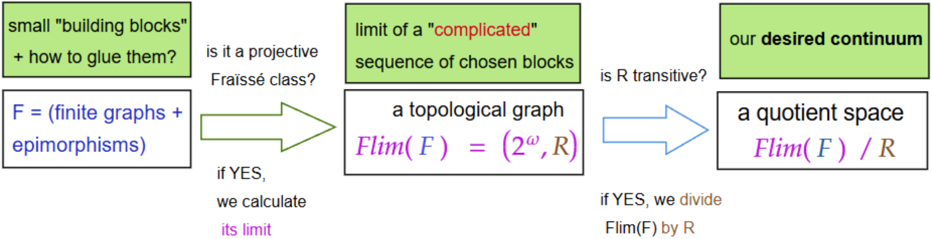
Theorem (T.Irwin, S.Solecki, [IS], 2006)

Let $\mathcal{F}$ be a projective Fraïssé category. There exists a unique (up to isomorphism) topological graph $\mathbb{F} \in \mathcal{F}^\omega$ so that:

- for each $A \in \mathcal{F}$, there is a morphism $\mathbb{F} \to A$
- for $A, B \in \mathcal{F}$, morphisms $f : \mathbb{F} \to A$ and $g : B \to A$, there is a morphism $h : \mathbb{F} \to B$ with $f = g \circ h$.

The structure $\mathbb{F}$ in Theorem above is called the projective Fraïssé limit of the projective Fraïssé category $\mathcal{F}$

How to produce continua using projective Fraisse?



small "building blocks"
+ how to glue them?

F = (finite graphs + epimorphisms)

is it a projective Fraïssé class?

if YES, we calculate its limit

limit of a "complicated" sequence of chosen blocks

a topological graph
$Flim(F) = (2^{\omega}, R)$

is R transitive?

if YES, we divide Flim(F) by R

our **desired continuum**

a quotient space
$Flim(F) / R$

How to build the universal Knaster continuum using projective Fraisse theory [I-1]

Objects in $\mathcal{K}$: paths (finite linear graphs) with distinguished endpoint

All pictures on this slide are taken from S.Iyer's presentation *Homeomorphism groups of Knaster continua*, available at: `https://logic.math.caltech.edu/slides/2022-11-30.pdf`

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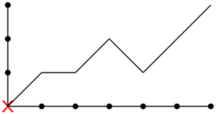
Epimorphisms between paths: surjective maps that preserve edge relation and distinguished endpoint (discrete analogues of continous surjections)



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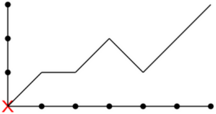
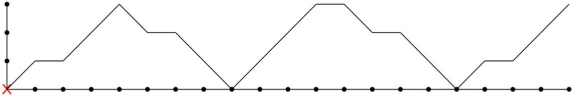


Morphisms in $\mathcal{K}$: those epimorphisms which are *piecewise monotone* (discrete analogues of confluent continous surjections)



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Results from [I-1]

Theorem, (S.Iyer, [I-1], 2022)

- $\mathcal{K}$ is a projective Fraïssé category.
- Let $\mathbb{K}$ be the Fraïssé limit of $\mathcal{K}$. Then the relation $R^{\mathbb{K}}$ is a closed equivalence relation and $\mathbb{K}/R^{\mathbb{K}}$ is homeomorphic to the universal Knaster continuum.

Theorem, (S.Iyer, [I-1], 2022)

A topological space X is a Knaster continuum if and only if X is homeomorphic to the quotient of some pre-continuum in $\mathcal{K}^{\omega}$

CHAPTER 3

The *discrete*, spectral incarnation of the Knaster continuum

with morphisms being relations $R^n_m : K_m \to K_n$,
for $K_n =:$ finite linear graph endowed with discrete topology

Motivation

Topological space $\implies$ poset of open sets

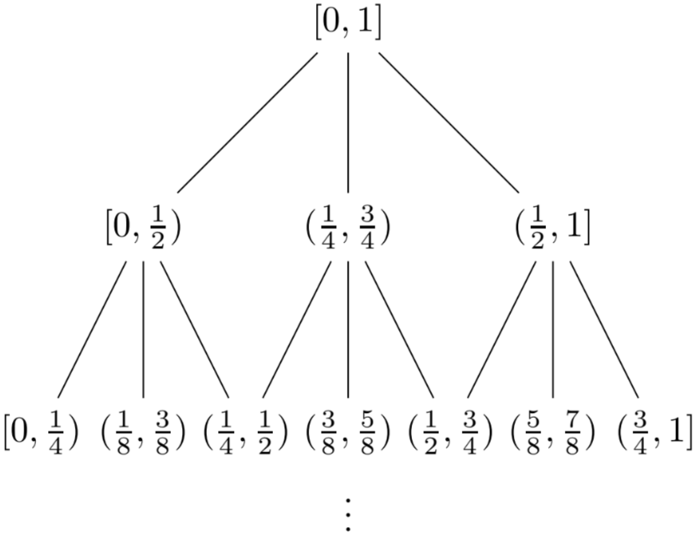
- Let X be a metrizable compact space.
- Then there exist a sequence $(\mathbb{P}_n)_{n\in\mathbb{N}}$, where
  - each $\mathbb{P}_n$ is a finite open cover of X,
  - for every n, $\mathbb{P}_n$ is refined by $\mathbb{P}_{n+1}$,
  - every open cover of X is refined by some $\mathbb{P}_n$.

  In particular, $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is a basis of X.
- Take the disjoint union $\mathbb{P} = \bigsqcup_{n\in\mathbb{N}} \mathbb{P}_n$ and order $\mathbb{P}$ by inclusion.

Poset of open sets $\implies$ topological space (Theorem, [BBV-1])

Every metrizable compact space X can be reconstructed from the poset structure of any countable basis $\{B_n : n \in \mathbb{N}\}$ such that $diam(B_n) \to 0$.

Example: the arc



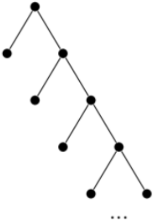
$[0,1]$

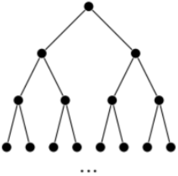
$[0,\frac{1}{2})$ $(\frac{1}{4},\frac{3}{4})$ $(\frac{1}{2},1]$

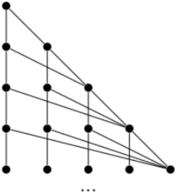
$[0,\frac{1}{4})$ $(\frac{1}{8},\frac{3}{8})$ $(\frac{1}{4},\frac{1}{2})$ $(\frac{3}{8},\frac{5}{8})$ $(\frac{1}{2},\frac{3}{4})$ $(\frac{5}{8},\frac{7}{8})$ $(\frac{3}{4},1]$

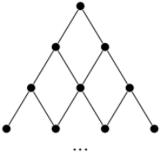
$\vdots$

More examples


convergent sequence


Cantor space 2^ω


cofinite topology on ω


discrete space $\{0,1\}$

What is a spectrum?

Let $\mathbb{P} = (\mathbb{P}_n, \sqsupset_n^m)$ be a ω-poset where $(\mathbb{P}_n)$ - levels of $\mathbb{P}$ and $(\sqsupset_n^m)$ - relations between levels of $\mathbb{P}$.

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- A **selector** is any upwards closed subset of $\mathbb{P}$ which intersects all levels of $\mathbb{P}$,
- A **spectrum** of $\mathbb{P}$, denoted $S\mathbb{P}$, is a space of minimal selectors wrt containment:

$$S\mathbb{P} = \{S \subseteq \mathbb{P} : S \text{ is a } \subseteq -\text{minimal selector}\}$$

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- For $p \in \mathbb{P}$ we define a subbasic open set

$$p^{\in} = \{S \in S\mathbb{P} : p \in S\}.$$

Sets $\{p^{\in} : p \in \mathbb{P}_n, n \in \mathbb{N}\}$ form a basis of topology on $S\mathbb{P}$.

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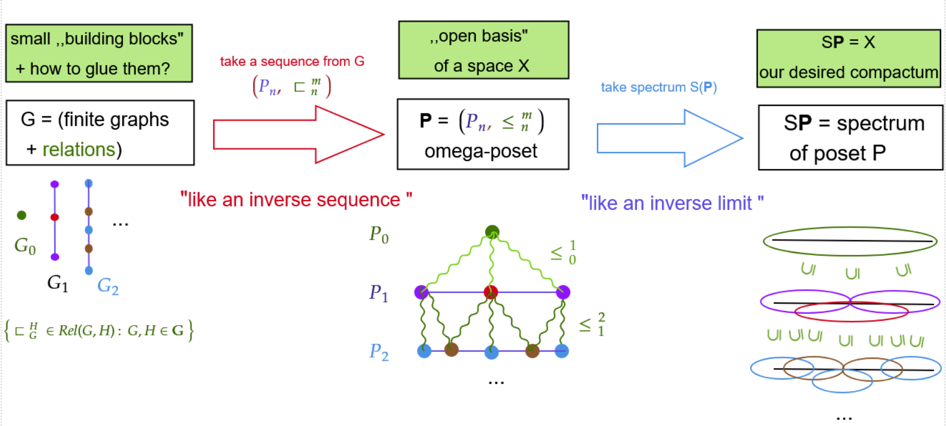
Intuition: Spectrum of a poset $\rightsquigarrow$ Stone space of a Boolean algebra.

Spectrum S vs inverse limit $\underleftarrow{\lim}$

inverse limit	**spectrum**
$\begin{tikzcd} & & \underleftarrow{lim}(X_i, f_i) \arrow[dll, "f_{0,\infty}"'] \arrow[dl, "f_{1,\infty}"] \arrow[d, "f_{2,\infty}"] \arrow[dr, "f_{3,\infty}"] & \dots \\ X_0 & X_1 \arrow[l, "f_1"'] & X_2 \arrow[l, "f_2"'] & \dots \arrow[l, "f_3"'] \end{tikzcd}$	$\begin{tikzcd} & & S\mathbb{P} \arrow[dll, "R_{0,\infty}"'] \arrow[dl, "R_{1,\infty}"] \arrow[d, "R_{2,\infty}"] \arrow[dr, "R_{3,\infty}"] & \dots \\ \mathbb{P}_0 & \mathbb{P}_1 \arrow[l, "R_0^1"'] & \mathbb{P}_2 \arrow[l, "R_1^2"'] & \dots \arrow[l, "R_2^3"'] \end{tikzcd}$
space of all threads	space of all minimal selectors
open sets are of the form $\left\{f^{-1}_{n,\infty}(U) : U \subseteq X_n,\ n \in \mathbb{N}\right\}$	open sets are of the form $\left\{R^{-1}_{n,\infty}(p) : p \in \mathbb{P}_n,\ n \in \mathbb{N}\right\}$
$x \in X_n \leftrightsquigarrow$ basic clopen set in $\underleftarrow{lim}(X_i, f_i)$ $x \in X_n \leftrightsquigarrow$ basic closed set in $\underleftarrow{lim}(X_i, f_i)/\sqcap$	$p \in \mathbb{P}_n \leftrightsquigarrow$ basic open set in $S\mathbb{P}$
every compact metrizable space can be built this way (but we have to take a quotient space)	every compact metrizable space can be built this way (no quotient needed)

How to produce continua using ω-posets and spectra?



small ,,building blocks"
+ how to glue them?

G = (finite graphs
+ relations)

$\left\{\sqsubset_G^H \in Rel(G,H) : G,H \in \mathbf{G}\right\}$

take a sequence from G
$\left(P_n, \sqsubset_n^m\right)$

"like an inverse sequence "

,,open basis"
of a space X

P = $\left(P_n, \leq_n^m\right)$
omega-poset

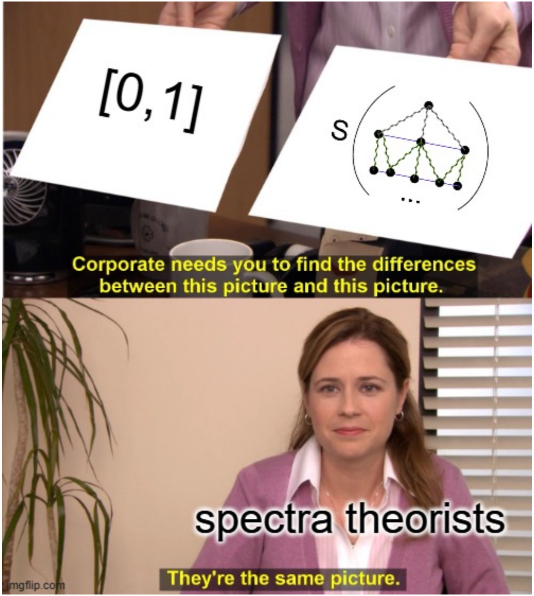
take spectrum S(**P**)

"like an inverse limit "

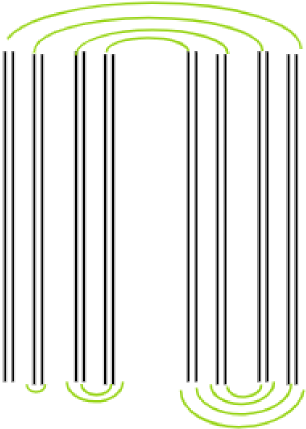
S**P** = X
our desired compactum

S**P** = spectrum
of poset P

They are the same picture



What about the Knaster continuum?



We define a Fraïssé category $\mathcal{K}_{rel}$ of *piecewise monotone relations between paths* then take a special sequence form $\mathcal{K}_{rel}$ and show that its induced poset gives the Knaster continuum K_2 as a spectrum.

What about the Knaster continuum?

Theorem (A. Bartoš, J.Ś)

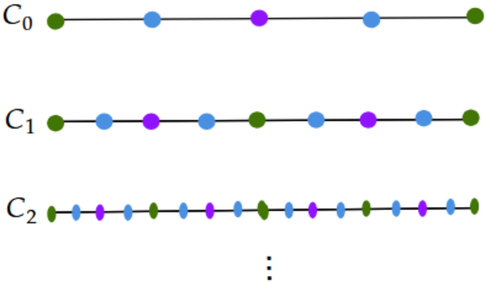
- Let $K = \varprojlim(I_n, \Delta)$, where $\Delta : I_{n+1} \to I_n$ is a tent map of degree 2.
- Let $\mathbb{D}_n = \pi_n^{-1}[C_{2n}]$, be a family of covers of K, where C_n is regular cover of I_n and $\pi_n : K \to I_n$ is a projection.
- Let $R_n^{n+1} : \mathbb{D}_{n+1} \to \mathbb{D}_n$ be a composition of tent pattern and double splitting pattern.
- Then $(\mathbb{D}_n, R_n^{n+1})$ is a Fraïssé sequence in $\mathcal{K}_{rel}$.
- Let $\mathbb{D}$ be an induced poset of a sequence $(\mathbb{D}_n, R_n^{n+1})$.
- Then

$$S\mathbb{D} \overset{\text{homeo}}{\simeq} K.$$

Knaster continuum as a spectrum - part 1

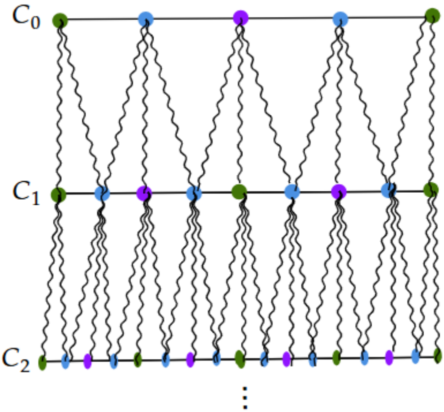
We will introduce a family of **regular covers** $(C_n)_{n\in\mathbb{N}}$ of $I = [0, 1]$.

Let C_n be a linear graph of length $2^{n+2} + 1$.



Knaster continuum as a spectrum - part 2

We define relations between finite sets C_n such that each element of C_n is related to 3 consecutive elements of C_{n+1} and consecutive pairs in C_n are related to exactly 1 common element in C_{n+1}.

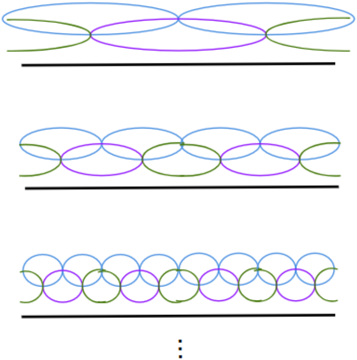


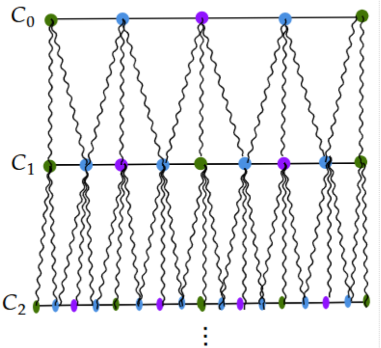
Knaster continuum as a spectrum - part 3

Definition: graphs $(C_n)_{n\geq 1}$ are **regular covers** of $[0,1]$.

Intuition:

- intersection of basic sets $\iff$ edge in graph
- containment of sets $\iff$ order relation in poset $C = \bigsqcup C_n$

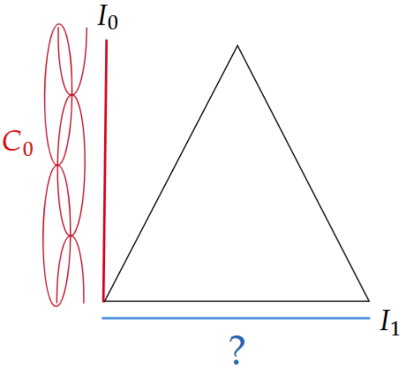




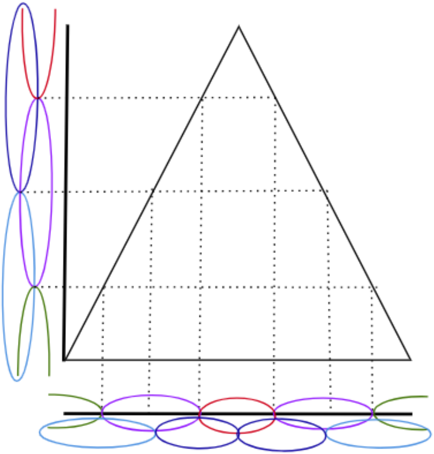
Knaster continuum as a spectrum - part 4

Take $\Delta : I_1 \rightarrow I_0$, *let* C_0 *be a cover of* I_0.

What will be the preimage $\Delta^{-1}[C_0]$?

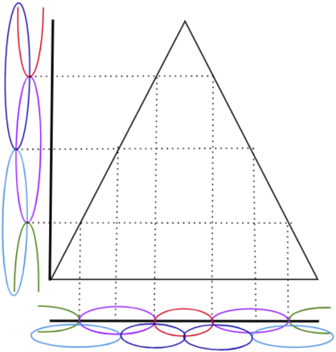


Knaster continuum as a spectrum - part 5



$$C_1 = \Delta^{-1}[C_0]$$

Knaster continuum as a spectrum - part 6



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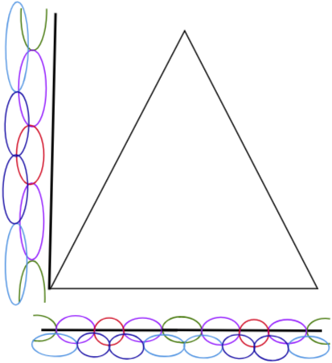
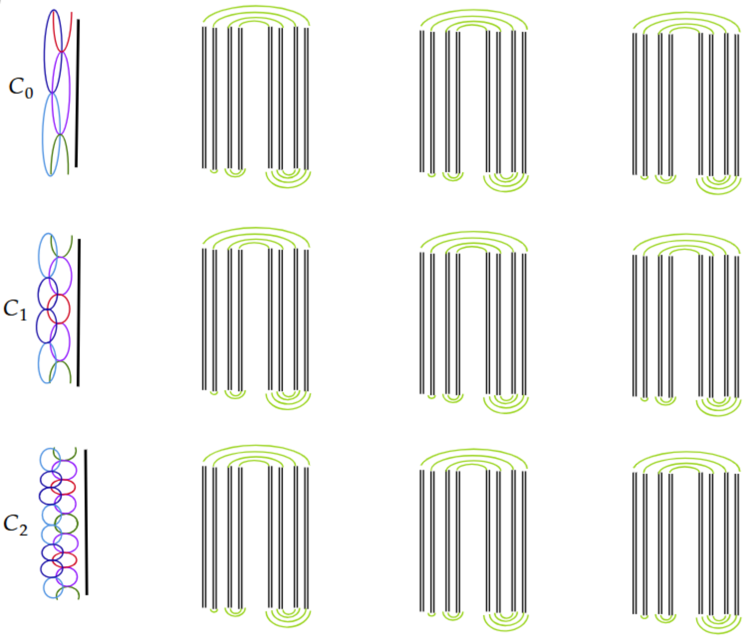


$$C_2 = \Delta^{-1}[C_1]$$

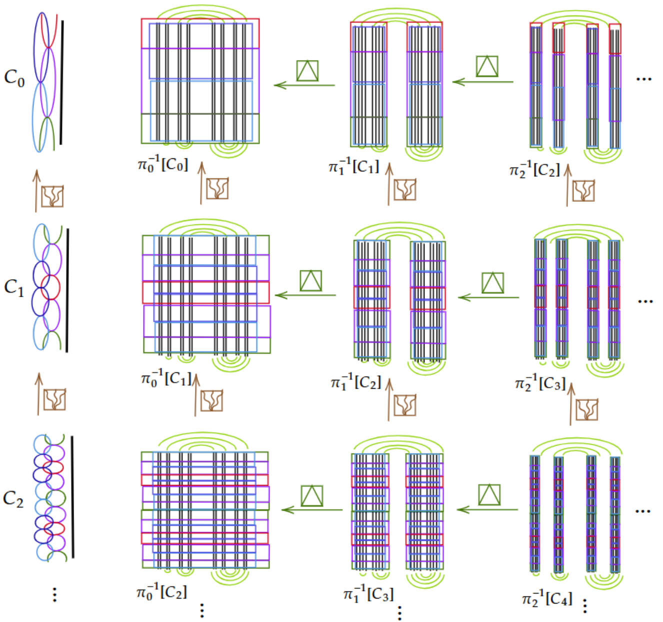
in general,

$$C_{n+1} = \Delta^{-1}[C_n]$$

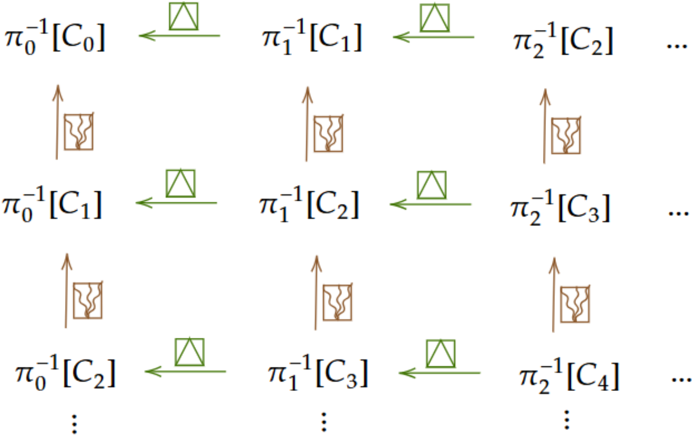
Knaster continuum as a spectrum - part 7



Knaster continuum as a spectrum - part 8

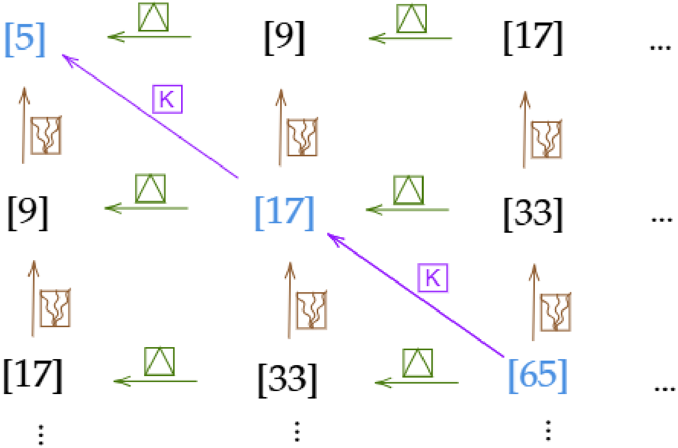


Knaster continuum as a spectrum - part 9

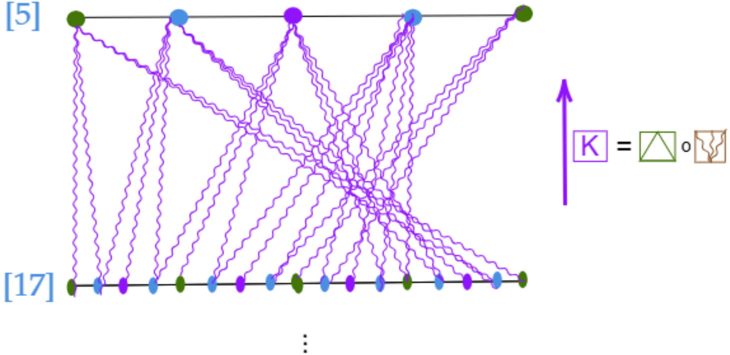


$$\begin{tikzcd}
\pi_0^{-1}[C_0] & \pi_1^{-1}[C_1] \arrow[l] & \pi_2^{-1}[C_2] \arrow[l] & \cdots \\
\pi_0^{-1}[C_1] \arrow[u] & \pi_1^{-1}[C_2] \arrow[l] \arrow[u] & \pi_2^{-1}[C_3] \arrow[l] \arrow[u] & \cdots \\
\pi_0^{-1}[C_2] \arrow[u] & \pi_1^{-1}[C_3] \arrow[l] \arrow[u] & \pi_2^{-1}[C_4] \arrow[l] \arrow[u] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

Knaster continuum as a spectrum - part 10



Knaster continuum as a spectrum - part 11



Main tool used in obtaining Knaster as a spectrum

Theorem (BBV-2)

Let X be a non-empty compact metric space and let (G_n) be a sequence of minimal open covers of X such that

1. G_n consolidates G_{n+1} for every $n \in \mathbb{N}$,
2. $G_n \cap G_{n+1} = \varnothing$ for every $n \in \mathbb{N}$,
3. $\lim_{n\to\infty} \max\{diam(g) : g \in G_n\} = 0$.

Endow every G_n with the edge relation

$$g \sqcap g' \iff g \cap g' \neq \varnothing,$$

and let, for $m \leq n$, $\sqsupset^m_n \subseteq G_m \times G_n$ be the restriction of the inclusion relation on $\mathcal{P}(X)$. Let $\mathbb{P}$ denote the poset $\bigcup_{n\in\mathbb{N}} G_n$ with inclusion as the order relation.

Then

$$S\mathbb{P} \overset{\text{homeo}}{\simeq} X$$

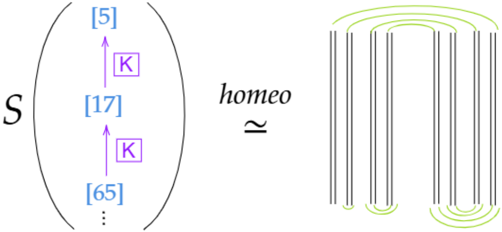
What's the recipe for the Knaster continuum?

Knaster continuum as a spectrum

Use this theorem for a sequence

$$\pi_n^{-1}(C_{2n})$$

where C_n is the $n-th$ regular cover of the interval $I=[0,1]$ and $\pi_n : K \to I_n$ is a projection.



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Děkuju!


propojení s Cantorovým prostorem
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nekonečné kapsy
pásek pro regulaci kardinality nekonečných kapes
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