Around the decomposability of Borel functions

Winter School in Abstract Analysis 2026
section Set Theory & Topology

Orazio Nicolosi

Dipartimento di matematica "Giuseppe Peano"

February 2nd, 2026

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Notation

We write $f \in \mathrm{dec}(\mathbf{\Sigma}^0_\alpha)$ *if* f *is decomposable as countable union of* $\mathbf{\Sigma}^0_\alpha$*-measurable functions (on their domains).*

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Theorem (Solecki Dichotomy)

Given X analytic space, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\mathbf{\Sigma}^0_1)$ or f "contains" the Pawlikowski *function $P : (\omega+1)^\omega \to \omega^\omega$, defined as:*

$$P(\eta)(n) = \begin{cases} 0 & \text{if } \eta(n) = \omega \\ \eta(n)+1 & \text{otherwise} \end{cases}$$

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This result was first proved by Solecki (1998) for Baire class 1 functions and then extended to all Borel function by Zapletal (2004) and by Pawlikowski and Sabok to all functions with analytic graphs (2012).

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Fact (Carroy, Lutz)

The Pawlikowski function is "equivalent" to the **Turing Jump** *i.e. the function*

$$\begin{aligned} J : \omega^\omega &\to 2^\omega \\ x \mapsto J(x) &= x' = \{e \in \omega \mid \varphi^x_e(e)\downarrow\} \end{aligned}$$

The Solecki Dichotomy (more precisely)

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f **embeds topologically** into g ($f \sqsubseteq g$) if there exist two topological embeddings $\varphi : X_f \to X_g$ and $\psi : Y_f \to Y_g$ such that $\psi \circ f = g \circ \varphi$.

$$\begin{array}{ccc} X_g & \xrightarrow{g} & Y_g \\ \uparrow\varphi & & \uparrow\psi \\ X_f & \xrightarrow{f} & Y_f \end{array}$$

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Different reducibilities: (strong) continuous reducibility

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Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **continuously reducible** to g ($f \leqslant_s g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \rightharpoonup Y_f$ such that $\forall x \in X_f (f(x) = \psi(g(\varphi(x))))$.

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The Solecki Dichotomy can be "weakened" with $\leqslant_s$ in place of $\sqsubseteq$ and is not difficult to prove that $J \equiv_s P$. Moreover, Marks and Montalbàn recently announced:

Theorem (Generalized Solecki Dichotomy)

Given $1 \leqslant \alpha < \omega_1$, *and* $f : \omega^\omega \to \omega^\omega$ *Borel; then either* $f \in \mathrm{dec}(\mathbf{\Sigma}^0_{<(1+\alpha)})$ *or* $J^{(\alpha)} \leqslant_s f$.

Different reducibilities: weak continuous reducibility

Definition

Given X_f, Y_f, X_g, Y_g topological spaces and functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is **weakly (continuously) reducible** to g ($f \leqslant_w g$) if there exist two partial continuous functions $\varphi : X_f \rightharpoonup X_g$ and $\psi : Y_g \times X_f \rightharpoonup Y_f$ such that $\forall\, x \in X_f (f(x) = \psi(g(\varphi(x)), x))$.

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Theorem (Lutz, Carroy, Nicolosi)

Given X Polish space, Y separable metrizable and $f : X \to Y$ Borel; then either $f \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$ or $J^{(n)} \leqslant_w f$.

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This result was originally proved by Lutz for functions on the Baire space.

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Definition

Let (X, d) be a separable metric space and $\boldsymbol{r} = (r_i)_{i\in\omega}$ an enumeration (possibly with repetitions) of a dense subset of X. We say that $\boldsymbol{r}$ is a **recursive presentation** of X if the relations on ω^3

$$P(i, j, k) \Leftrightarrow \mathrm{d}(r_i, r_j) \leqslant q_k$$
$$Q(i, j, k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k$$

are recursive. The structure $(X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented metric space**. If moreover (X, d) is complete, then $(X, \mathrm{d}, \boldsymbol{r})$ is called **recursively presented Polish space**.

Basic spaces

Definition

A **basic space** $\mathcal{X}$ is a pair $(X, (V_n)_{n\in\omega})$ where X is a second countable topological space, $(V_n)_{n\in\omega}$ is an enumeration (possibly with repetitions) of a countable basis of the topology of X[a], and there is a semirecursive relation $R \subseteq \omega^3$ such that:

$$V_m \cap V_n = \bigcup_{p\in\omega} \{V_p \mid R(m,n,p)\}$$

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Fact

Subspaces, finite products and countable products of basic spaces are basic spaces.

Σ^0_1 sets

Definition

A subset A of a basic space $\mathcal{X}$ is called $\Sigma^0_1(\mathcal{X})$ (also said **effectively open** in $\mathcal{X}$) if there is a semirecursive set A^* in ω such that

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Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, a function $f : \mathcal{X} \to \mathcal{Y}$ is Σ^0_1-**recursive** if its diagram is Σ^0_1, that is:

$$D_f = \{(x, n) \mid f(x) \in V_n^{\mathcal{Y}}\} \in \Sigma^0_1(\mathcal{X} \times \omega)$$

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Any Σ^0_1-recursive function is also continuous. Moreover, the Σ^0_1-recursive functions generalize not only the computable functions on the natural numbers but also the computable functions on ω^ω (Type-2 theory of effectivity).

Recursive spaces

Definition

Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces, $f : \mathcal{X} \to \mathcal{Y}$ is a **recursive isomorphism** if it is Σ^0_1-recursive, bijective, and has Σ^0_1-recursive inverse.

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Fact

Any recursively presented metric space $(X, \mathrm{d}, \boldsymbol{r})$ *admits a structure of basic space by considering* $V_n = \{x \in X \mid \mathrm{d}(\boldsymbol{r}_{(n)_0}, x) < q_{(n)_1}\}$.

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Definition

A basic space $\mathcal{X}$ is **recursive** if it is recursively isomorphic to a subspace of a recursively presented metric space.

Proposition

Any recursive space is recursively isomorphic to a subspace of the Hilbert cube $[0,1]^\omega$.

Lightface pointclasses and universal sets

The pointclasses in the **Arithmetical Hierarchy** are defined (by induction) as:

$$\Sigma^0_1 \qquad \Pi^0_1 = \neg\Sigma^0_1 \qquad \Sigma^0_{n+1} = \exists^0 \Pi^0_n \qquad \Pi^0_{n+1} = \neg\Sigma^0_{n+1} \qquad \Delta^0_n = \Sigma^0_n \cap \Pi^0_n$$

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Definition

Given $\mathcal{X}$, $\mathcal{Y}$ basic spaces and Γ lightface pointclass, $G \in \Gamma(\mathcal{X} \times \mathcal{Y})$ is **universal** for $\Gamma(\mathcal{Y})$ if

$$\forall P \subseteq Y(P \in \Gamma \Leftrightarrow \exists x \in X(P = G_x))$$

where $G_x = \{y \in Y \mid G(x, y)\}$ is called x-section of G.

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Moreover, the process of relativization can be extended also to these pointclasses, obtaining this way the corresponding *topological/boldface pointclasses*.
In particular, for any recursive space $\mathcal{X}$: $\boldsymbol{\Sigma}^0_n(X) = \bigcup_{\alpha \in \omega^\omega} \Sigma^{0,\alpha}_n(\mathcal{X})$.

Continuous degrees

Definition

Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, $y \in Y$ is **representation reducible** to $x \in X$ (and we write $y \leqslant_M x$) if $y = f(x)$, for some $f : \mathcal{X} \rightharpoonup \mathcal{Y}$ is a partial Σ^0_1-recursive function on its domain.

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Definition (Continuous degrees [Mil04] and [GKN20])

Given $\mathcal{X}$ recursive space, the **continuous degree** of $x \in \mathcal{X}$ is its equivalence class under the relation $\equiv_M$ (over elements of recursive spaces).

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Given $\mathcal{X}$ recursive space, the **continuous degree** of $x \in \mathcal{X}$ is its equivalence class under the relation $\equiv_M$ (over elements of recursive spaces).

Fact

Given $\mathcal{X}$ recursive space,

1. $\forall\, x \in X\; \exists\, z \in [0,1]^\omega (x \equiv_M z)$.
2. $\forall\, x, y \in \omega^\omega (y \leqslant_T x \Leftrightarrow y \leqslant_M x)$

The Turing jump for recursive spaces

The notion of Turing Jump can be extended to all recursive spaces

Definition ([GKN20])

Given $\mathcal{X}$ recursive space, the $\Sigma^{0,\alpha}_n$**-jump** is the function $J^{(n),\alpha}_{\mathcal{X}} : \mathcal{X} \to 2^\omega$ defined as:

$$J^{(n),\alpha}_{\mathcal{X}}(x) = \{e \in \omega \mid x \in H^{\mathcal{X}}_{\Sigma^0_n,\alpha,e}\}$$

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Theorem (Generalized Posner-Robinson for recursive spaces)

Given $\mathcal{X}$ and $\mathcal{Y}$ recursive spaces, for every $n \in \omega$
$\forall\, x \in X\ \forall\, y \in Y(y \leqslant_M x^{(n)}\ \dot\vee\ \exists\, g \in 2^\omega(x \oplus y \oplus g \geqslant_M (g \oplus x)^{(n+1)})$.

A modification of Lutz's game

Given two functions $f : \omega^\omega \to \omega^\omega$ and $g : \mathcal{X} \to \mathcal{Y}$ (where $\mathcal{X}$ and $\mathcal{Y}$ are recursive spaces) we define the game $G_M(f, g)$ as follows:

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Player 1		x_0		x_1		...
Player 2	e		b_0, z_0		b_1, z_1	...

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Player 1		x_0	x_1	$\ldots$
Player 2	e	$b_0,\ z_0$	$b_1,\ z_1$	$\ldots$

Player 2 wins if and only if b is a $\rho_{\mathcal{X}}$-name for an element $y \in \mathcal{X}_g$ (i.e. $b \in \mathrm{dom}(\rho_{\mathcal{X}})$) and $f(x) = \Phi_e^{(\mathcal{Y}\times\omega^\omega\times\omega^\omega),\omega^\omega}(g(\rho_{\mathcal{X}}(b)), x, z)$, where $\rho_{\mathcal{X}}:\omega^\omega \rightharpoonup X$ is the *admissible representation* defined as: $\rho_{\mathcal{X}}(p) = x \Leftrightarrow \mathrm{ran}(p) = \{n \in \omega \mid x \in V_n^{\mathcal{X}}\}$

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Remark

If f and g are Borel and the domain of $\rho_\mathcal{X}$ is Borel, then so is the payoff of $G_M(f, g)$.

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Lemma (Lutz, Carroy, Nicolosi)

- *If Player 2 has a winning strategy for $G_M(f,g)$, then $f\leqslant_w g$.*
- *If Player 1 has a winning strategy for $G_M(J^{(n)},g)$, then $g\in\operatorname{dec}(\boldsymbol{\Sigma}^0_n)$.*

Using Borel determinacy we get

Theorem (Lutz, Carroy, Nicolosi)

Given $\mathcal{X}$, $\mathcal{Y}$ recursive spaces such that $\rho_{\mathcal{X}}$ has Borel domain and $g : \mathcal{X} \to \mathcal{Y}$ Borel; then either $g \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$ or $J^{(n)} \leqslant_w g$.

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The domain of the considered representation is Borel for any recursively presented Polish spaces.

Thus we get:

Corollary

Given $\mathcal{X}$ recursively presented Polish space, $\mathcal{Y}$ recursive space and $g : X \to Y$ Borel; then either $g \in \mathrm{dec}(\boldsymbol{\Sigma}^0_n)$ or $J^{(n)} \leqslant_w g$.

Further directions

- Is it possible to find a game that characterizes weak reducibility $\leqslant_w$ in a wider context? (e.g. in separable metrizable spaces)

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Thank you for your attention.

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