Weakly Compact Cardinals in the Bristol Model

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Motivating Questions

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The Bristol model acts as an excellent source of counterexamples!

Symmetric Extensions I

Take a 'base model' $V \models \text{ZFC}$, and a forcing extension $V[G]$. The forcing extension must also be a model of choice, but we can define an *intermediate model*, M, i.e.

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$$V \subseteq M \subseteq V[G],$$

where M is a model of ZF which may witness a failure of choice. We define a symmetric extension using a symmetric system.

Definition

A *symmetric system* $\mathscr{S}$ is a triple $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$, where $\mathbb{P}$ is a forcing notion, $\mathscr{G} \subseteq \mathrm{Aut}(\mathbb{P})$ is a group of automorphisms of $\mathbb{P}$, and $\mathscr{F}$ is a normal filter of subgroups on $\mathscr{G}$.

Symmetric Extensions II

Let $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ be a symmetric system and let $\dot{x}$ be a $\mathbb{P}$-name. We say that $\dot{x}$ is symmetric if $\mathrm{sym}_{\mathscr{G}}(\dot{x}) = \{\pi \in \mathscr{G} \mid \pi\dot{x} = \dot{x}\} \in \mathscr{F}$, i.e the group of automorphisms which fix $\dot{x}$ is in the filter.
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Theorem

Suppose that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a symmetric system, and $G \subseteq \mathbb{P}$ is a V-generic filter. Then $\mathrm{HS}^G = \{\dot{x}^G \mid \dot{x} \in \mathrm{HS}\}$ is a transitive subclass of $V[G]$ which contains V and satisfies the axioms of ZF.

We refer to this HS^G as a symmetric extension of V.

Small Violations of Choice

Definition

We say that *small violations of choice* holds in a model if there is a set S such that for all X there is an ordinal η and a surjection $f : S \times \eta \to X$.

Theorem (Blass, Usuba)

The following are equivalent:

- $M \models \mathsf{SVC}$.
- *We can restore choice using a set forcing on M.*
- *There is an inner model $V \subseteq M$ such that $V \models \mathsf{ZFC}$ and M is a symmetric extension of V.*
- *There is an inner model $V \subseteq M$ such that $V \models \mathsf{ZFC}$ and there is $x \in M$ such that $M = V(x)$.*

Kinna–Wagner Principles

We use $\mathcal{P}^\alpha(x)$ to denote taking the αth iterated power set of x.

Definition

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Observe that $\mathsf{KWP}_n \to \mathsf{KWP}_{n+1}$ and that KWP_0 is equivalent to the axiom of choice. We can think of Kinna–Wagner rank as capturing 'how far away from being well-orderable' a set is.

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Proposition

SVC $\to$ KWP, *but the reverse does not hold.*

New choiceless intermediate models

Question

Starting with a model of ZFC, *are there intermediate models of* ZF *which are not models of* SVC*?*

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To generate a model with unbounded Kinna–Wagner degree, we will need a new approach - we will use a *class length iteration of symmetric extensions*.

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Proposition (Karagila)

If $\beta > \alpha$*, then* $M_\beta \models \neg\mathrm{KWP}_\alpha$*.*

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Note that a generalisation of the Bristol model (intermediate models of ‘deep failure of choice’) has since been defined by Hayut and Shani. Their construction can be done over any ground model.

Which large cardinals can live in the Bristol model?

We defined the Bristol model assuming the ground model is a model of L, which quickly limits which large cardinals can exist an a (class) symmetric extension. But, the Bristol model construction still works under weaker assumptions, namely, we need to assume GCH and $\square^*_\lambda$ for singular λ. These assumptions are compatible with all known large cardinals with canonical inner models.

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Corollary

If κ is a large cardinal defined by the existance of a set of ordinals, and this largeness is preserved by adding a Cohen real, then κ remains large in the Bristol model.

So, if we construct the Bristol model over a ground model of '$L +$ there exists a Mahlo cardinal κ', then κ will remain Mahlo in the Bristol model.

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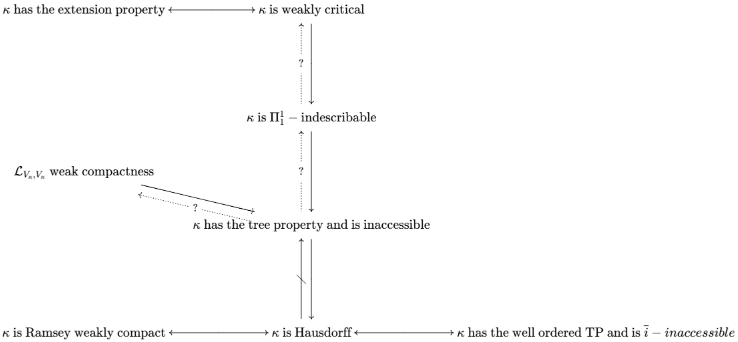

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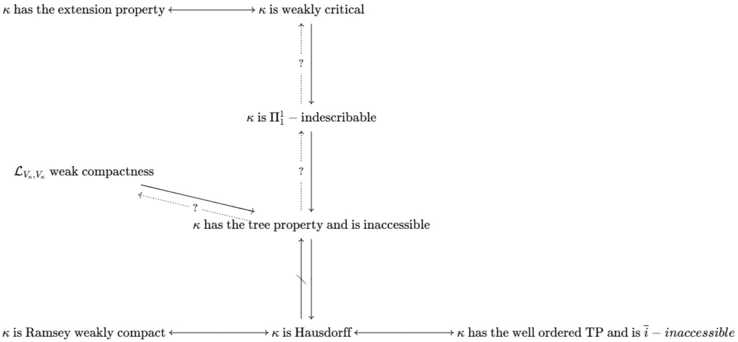

(Showing that weak criticality is equivalent to the extension property and implies indescribability is due to Hayut and Karagila. Other implications are ongoing work of myself and Lyra Gardiner.)

The tree property

We say κ has the tree property if every κ-tree has a branch. In ZFC, κ having the tree property and being inaccessible is equivalent to being weakly compact. We need to be more careful with our definitions in ZF, and we also want to assume that our trees live in V_κ.

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Proposition (D, Karagila)

Let κ be a weakly compact cardinal, then there is a symmetric extension in which every tree of height κ with levels of size $< \kappa$ has a branch, and for any $\alpha < \kappa$, we have that $\kappa \nleq 2^\alpha$. We can show 2^ω surjects onto κ.

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Proposition (D, Karagila)

If κ is weakly compact, then there is a symmetric extension in which κ is honestly inaccessible, every T of height κ with levels $< \kappa$ has a branch, but there is a tree $T \subseteq V_\kappa$ of height κ such that $\kappa \not\leq T_\alpha$ for all $\alpha < \kappa$, and T does not have a branch.

The tree property in the Bristol model

Note that if κ is inaccessible in the ground model, κ will be honestly inaccessible in the Bristol model.

Definition

We say that T is a (κ, α)-*tree* if T is a tree of height T such that $T_\alpha \in V_\kappa$ and T has Kinna–Wagner rank $\leqslant \alpha$.

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Theorem (D, Karagila)

Assume κ is weakly compact. In M_κ, for $\alpha < \kappa$, every (κ, α)-tree has a branch. However, there is a (κ, κ)-tree which does not have a branch.

Proof.

(Sketch). For $\alpha < \kappa$, every α-set in M_κ has already been added in $M_{\alpha+1}$. M_α is generated by a symmetric extension of size $< \kappa$. Being weakly critical is preserved by small forcings, and being weakly critical implies the tree property, so (κ, α)-trees must have branches. However, we can build a (κ, κ)-tree in the forcing $\mathbb{Q}_\kappa$, and argue by homogeneity that it does not have a branch. □

Future work

Conjecture (D, Karagila)

In $M_{\kappa+1}$, κ *has the tree property. That is, every* (κ, κ)*-tree has a branch.*

Question

If κ *is weakly critical in L, is* κ *weakly critical in the Bristol model?*

Conjecture (D, Hayut, Karagila)

If κ *is weakly compact, then in the Hayut–Shani construction of models with deep failure of choice,* κ *is weakly critical.*

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What is the relationship between the tree property and weakly critical cardinals? We conjecture being weakly critical is stronger than "inaccessible and has the tree property" without choice.

Summary

- Symmetric extensions are a generalisation of forcing, used to generate models with 'small' failures of choice.
- We can use class length iterations of symmetric extensions to define models with 'deeper' failures of choice, such as the Bristol model. These models may be an interesting source of counterexamples we can use to seperate definitions of large cardinals without choice.
- We can separate many different definitions of weak compactness without choice.
- We have to be very careful to define weak compactness using the tree property without choice in a 'strong' way.
- The tree property interacts with the Bristol model iteration in a subtle way, and there is more to understand about what large cardinals are preserved in the full Bristol model.

Workshop announcement!

Women, Lesbians, Intersex, Non-binary, Trans, Agender.
Frauen, Lesben, Intergeschlechtliche, Nichtbinäre, Trans, Agender.

FLINTA* — in Set Theory

Erwin Schrödinger Institute, Vienna

Invited Speakers

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Lyra **Gardiner** (Cambridge)
Victoria **Gitman** (CUNY)
Shoshana **Friedman** (CUNY)
Juliette **Kennedy** (Helsinki)
Aleksandra **Kwiatkowska** (Münster)
Heike **Mildenberger** (Freiburg)
Julia **Millhouse** (Vienna)
Sandra **Müller** (TU Wien)
Elena **Pozzan** (Turin)
Calliope **Ryan-Smith**
Catalina **Torres** (Barcelona)
Allison **Wang** (CMU)

FLINTA in Set Theory is the first conference about emphasizing the female and non-binary contribution to set theory. Our mission is to foster a more inclusive and diverse set theory community. All those who lend their support to our goals are invited to participate in this conference.*

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9th - 11th Sept 2026

ESI Erwin Schrödinger International Institute for Mathematics and Physics

TU WIEN fem*MA MATHEMATIKERINNEN

FWF Österreichischer Wissenschaftsfonds

Thank you!

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