Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

Rafał Filipów

Winter School in Abstract Analysis (2026)

section Set Theory & Topology

Jan 31-Feb 7, 2026

in Hejnice

Quasinormal convergence of sequences of functions

$f_n : X \to \mathbb{R}$ – sequence of continuous functions ($n \in \omega$)

Pointwise convergence

$(f_n) \xrightarrow{P} f$ if $\forall \varepsilon > 0 \ \ \forall x \in X\, (\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathrm{Fin})$

Quasinormal convergence

$(f_n) \xrightarrow{QN} f$ if $\exists (\varepsilon_n) \to 0^+ \, \forall x \in X\, (\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathrm{Fin})$

Relationship between pointwise and quasinormal convergence

- $(f_n) \xrightarrow{QN} f \implies (f_n) \xrightarrow{P} f$ [always "yes"]
- $(f_n) \xrightarrow{QN} f \not\Longleftarrow (f_n) \xrightarrow{P} f$ [in general "no"]

QN-spaces

- $(f_n) \xrightarrow{QN} f \quad \Longrightarrow \quad (f_n) \xrightarrow{P} f$ [always "yes"]
- $(f_n) \xrightarrow{QN} f \quad \not\Longleftarrow \quad (f_n) \xrightarrow{P} f$ [in general "no"]

Definition (Bukovský-Recław-Repický, 1991)

X is a QN-space if X does not distinguishes *pointwise* and *quasinormal* convergence i.e. for any sequence $f_n : X \to \mathbb{R}$ of continuous functions,

$$(f_n) \xrightarrow{P} f \quad \Longrightarrow \quad (f_n) \xrightarrow{QN} f.$$

Definition (Bukovský-Recław-Repický, 1991)

$$\text{non(QN-space)} = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a QN-space}\}$$

Purely combinatorial characterization of non(QN-space)

$$\mathrm{non}(\text{QN-space}) = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a QN-space}\}$$

Theorem (Bukovska ($\geq$); Bukovský-Recław-Repický ($\leq$); 1991)

$$\mathrm{non}(\text{QN-space}) = \mathfrak{b}$$

$$\mathfrak{b} = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \neg\Big(\exists g \in \omega^\omega \, \forall f \in \mathcal{F} \, (f \leq^* g)\Big)\right\}$$

where $f \leq^* g \iff \{n : \neg(f(n) \leq g(n))\} \in \mathrm{Fin}$.

Ideal convergence of sequences of functions

Let $\mathcal{I}$ be an ideal on ω.

Ideal pointwise convergence

$(f_n) \xrightarrow{P} f$ if $\forall \varepsilon > 0 \, \forall x \in X \, (\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathbf{Fin})$

$(f_n) \xrightarrow{\mathcal{I}-P} f$ if $\forall \varepsilon > 0 \, \forall x \in X \, (\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathcal{I})$

Ideal quasi-normal convergence

$(f_n) \xrightarrow{QN} f$ if $\exists (\varepsilon_n) \to 0 \, \forall x \in X \, (\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathbf{Fin})$

$(f_n) \xrightarrow{\mathcal{I}-QN} f$ if $\exists (\varepsilon_n) \xrightarrow{\mathcal{I}} 0 \, \forall x \in X \, (\{n : |f_n(x) - f(x)| \geq \varepsilon_n\} \in \mathcal{I})$

---

where

$(\varepsilon_n) \to 0$ if $\forall \varepsilon > 0 \, (\{n : |\varepsilon_n| \geq \varepsilon\} \in \mathbf{Fin})$.

$(\varepsilon_n) \xrightarrow{\mathcal{I}} 0$ if $\forall \varepsilon > 0 \, (\{n : |\varepsilon_n| \geq \varepsilon\} \in \mathcal{I})$.

$\mathcal{I}$-QN-spaces and non($\mathcal{I}$-QN space)

Definition

X is a $\mathcal{I}$-QN-space if for any continuous functions $f, f_n : X \to \mathbb{R}$,

$$(f_n) \xrightarrow{\mathcal{I}-P} f \quad \Longrightarrow \quad (f_n) \xrightarrow{\mathcal{I}-QN} f.$$

Definition

$$\text{non}(\mathcal{I}\text{-QN-space}) = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a } \mathcal{I}\text{-QN-space}\}$$

Theorem (The first guess is wrong)

$$\text{non}(\mathcal{I}\text{-QN-space}) \neq \mathfrak{b}(\mathcal{I})$$

$$\mathfrak{b}(\mathcal{I}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \neg\Big(\exists g \in \omega^\omega \, \forall f \in \mathcal{F} \, (f \leq^{\mathcal{I}} g)\Big)\right\}$$

where $f \leq^{\mathcal{I}} g \iff \{n : \neg(f(n) \leq g(n))\} \in \mathcal{I}$.

Reverse the order

Definition (How about this "be" number)

$$\mathfrak{b}(\geq^{\mathcal{I}}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \neg\Big(\exists g \in \omega^\omega\, \forall f \in \mathcal{F}\, (f \geq^{\mathcal{I}} g)\Big)\right\}$$

It does not work, since every family $\mathcal{F}$ is $\geq^{\mathcal{I}}$-bounded by the constant zero function.

Definition (let's restrict the field of the relation $\geq^{\mathcal{I}}$)

$$\mathcal{D}_{\mathcal{I}} = \{f \in \omega^\omega : f^{-1}(n) \in \mathcal{I} \text{ for every } n\}$$

Definition (This "be" works)

$$\mathfrak{b}_{QN}(\mathcal{I}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{D}_{\mathcal{I}} \wedge \neg\Big(\exists g \in \mathcal{D}_{\mathcal{I}}\, \forall f \in \mathcal{F}\, (f \geq^{\mathcal{I}} g)\Big)\right\}$$

Theorem (F.-Staniszewski-Kwela, and independently, Repický)

$$\text{non}(\mathcal{I}\text{-QN-space}) = \mathfrak{b}_{QN}(\mathcal{I})$$

Rothberger gaps

Definition

(ω, λ)-gap in a poset $(P, \leq)$ is a pair $(\mathcal{A}, \mathcal{B})$ of subsets of P such that

- $|\mathcal{A}| = \omega$ and $|\mathcal{B}| = \lambda$,
- $a \leq b$ for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$,
- there is no element $c \in P$ with $a < c < b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

Theorem (Rothberger, 1941)

$$\mathfrak{b} = \min\{\lambda : \exists\, (\omega, \lambda)\text{-gap in } (\mathcal{P}(\mathbb{N}), \subseteq^*)\}$$

i.e.

- there exists $(\omega, \mathfrak{b})$-gap in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$,
- and there is no (ω, λ) gap for $\lambda < \mathfrak{b}$.

The Rothberger numbers

$$\mathfrak{b} = \min\{\lambda : \exists\,(\omega,\lambda)\text{-gap in } (\mathcal{P}(\mathbb{N}), \subseteq^*)\}$$

Definition (Brendle-Mejía, 2014)

$$\mathfrak{b}_R(\mathcal{I}) = \min\left\{\lambda : \exists\,(\omega,\lambda)\text{-gap in } (\mathcal{P}(\mathbb{N}), \subseteq^{\mathcal{I}})\right\},$$

where

$$A \subseteq^{\mathcal{I}} B \iff A \setminus B \in \mathcal{I}.$$

- (Rothberger) $\mathfrak{b}_R(\mathrm{Fin}) = \mathfrak{b}$
- With the convention $\min\emptyset = \infty$, if $\mathcal{I}$ is a maximal ideal, $\mathfrak{b}_R(\mathcal{I}) = \infty$.

Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

Theorem (F.-Kwela)

$$\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{b}_R(\mathcal{I})$$

Question: Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

$$b_{QN}(\mathcal{I}) \stackrel{?}{=} \mathfrak{b}_R(\mathcal{I})$$

Silly answer

In general, "two $\mathfrak{b}$", because if $\mathcal{I}$ is a maximal ideal,

$$\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{c} < \infty = \mathfrak{b}_R(\mathcal{I}).$$

Theorem (F.-Kwela)

Consistently "two $\mathfrak{b}$" i.e.

consistently, there is an ideal $\mathcal{I}$ such that

$$\mathfrak{b}_{QN}(\mathcal{I}) < \mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{c}$$

Two $\mathfrak{b}$ or not two $\mathfrak{b}$ in Borel realm?

Question

$$\mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{c} \quad \text{for Borel ideals?}$$

Question

$$\mathfrak{b}_{QN}(\mathcal{I}) \stackrel{?}{=} \mathfrak{b}_R(\mathcal{I}) \quad \text{for Borel ideals with } \mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{c}?$$

Guessing

All examples suggest: not two $\mathfrak{b}$.

Bounds for some classes of ideals

Theorem

$\aleph_1 \leq \mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{c}$ and $\aleph_1 \leq \mathfrak{b}_R(\mathcal{I}) \leq \infty$

Theorem (F.-Kwela)

- $\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{d}$ for weak P-ideals
- $\mathfrak{b} \leq \mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{d}$ for P-ideals

Theorem (Kwela)

- Consistently, $\exists \mathcal{I}\,(\mathfrak{b}_{QN}(\mathcal{I}) > \mathfrak{d})$
- $\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{d}$ for Borel ideals

Theorem

- $\mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{b}$ for F_σ ideals (Todorčević)
- $\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{b}$ for Borel weak P-ideals (F.-Kwela)

  In particular, $\mathfrak{b}_{QN}(\mathcal{I}) \leq \mathfrak{b}$ for all Π^0_4 ideals.

Values for some ideals

Theorem

- $\mathfrak{b}_R(\mathcal{I}) = \mathfrak{b}$ for analytic P-ideals (Brendle-Mejía)
- $\mathfrak{b}_{QN}(\mathcal{I}) = \mathfrak{b}_R(\mathcal{I}) = \mathfrak{b}$ for meager P-ideals (F.-Kwela)

Theorem (F.-Kwela)

$\mathfrak{b}_{QN}(\mathrm{Fin}^n) = \mathfrak{b}_R(\mathrm{Fin}^n) = \mathfrak{b}$ for $n \in \omega \setminus \{0\}$

Theorem (F.-Kwela)

$\mathfrak{b}_{QN}(\mathcal{I}) = \aleph_1$ if $\mathcal{I}$ is generated by a MAD family or $\mathcal{I} = \mathcal{ED}$ or $\mathcal{I} = \mathcal{S}$ or $\mathcal{I} = \mathcal{I}_b$.

Theorem (Brendle-Mejía)

$\mathfrak{b}_R(\mathcal{I}) = \aleph_1$ if $\mathcal{I}$ is an F_σ ideal which is fragmented but not gradually fragmented (e.g. $\mathcal{I} = \mathcal{ED}_{\mathrm{Fin}}$).

Maximal ideals and some consequences

Theorem (Canjar, 1989)

Consistently, for each $\aleph_1 \leq \kappa, \lambda \leq \mathfrak{c}$ there are maximal ideals $\mathcal{I}$ and $\mathcal{J}$ with

$$\mathfrak{b}_{QN}(\mathcal{I}) = \kappa \quad \text{and} \quad \mathfrak{b}_{QN}(\mathcal{J}) = \lambda.$$

Theorem (F.-Kwela)

Consistently, there are ideals $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ with

$$\mathfrak{b} = \mathfrak{b}_{QN}(\mathcal{I}_0) < \mathfrak{b}_R(\mathcal{I}_0) \leq \mathfrak{c} \quad \text{and} \quad \mathfrak{b} < \mathfrak{b}_{QN}(\mathcal{I}_1) < \mathfrak{b}_R(\mathcal{I}_1) \leq \mathfrak{c}$$

$$\mathfrak{b} < \mathfrak{b}_{QN}(\mathcal{I}_2) = \mathfrak{b}_R(\mathcal{I}_2) \leq \mathfrak{d} = \mathfrak{c} \quad \text{and} \quad \mathfrak{d} < \mathfrak{b}_{QN}(\mathcal{I}_3) = \mathfrak{b}_R(\mathcal{I}_3) \leq \mathfrak{c}$$

Topologically defined ideals

Theorem

- $\mathfrak{b}_{QN}(\text{conv}) = \mathfrak{b}_{QN}(\text{ctbl}) = \aleph_1$ (Kwela)
- $\mathfrak{b}_{R}(\text{conv}) = \mathfrak{b}_{R}(\text{ctbl}) = \aleph_1$ (F.-Kwela)

In fact, if $\text{conv} \subseteq \mathcal{I} \subseteq \text{ctbl}$, then

$$\mathfrak{b}_{QN}(\mathcal{I}) = \mathfrak{b}_{R}(\mathcal{I}) = \aleph_1.$$

In particular, for each $2 \leq \alpha < \omega_1$,

$$\mathfrak{b}_{QN}(CB_\alpha) = \mathfrak{b}_{R}(CB_\alpha) = \aleph_1.$$

Ideals defined in category and measure manner

Theorem

- $\mathfrak{b}_R(\mathrm{nwd}) = \mathrm{add}(\mathcal{M})$ (Kankaanpää)
- $\mathfrak{b}_{QN}(\mathrm{nwd}) = \mathrm{add}(\mathcal{M})$ (Kwela)

Theorem (Kwela)

$\mathfrak{p} \leq \mathfrak{b}_{QN}(\mathrm{null}) \leq \mathrm{add}(\mathcal{M})$

Consequently, $\mathfrak{p} \leq \mathfrak{b}_R(\mathrm{null})$.
In particular, consistently, $\mathfrak{b}_R(\mathrm{null}) \neq \mathrm{add}(\mathcal{N})$.

Theorem (F.-Brendle-Cieślak-Kwela)

$\mathfrak{b}_{QN}(\mathrm{null}) = \mathfrak{b}_R(\mathrm{null}) = \mathrm{add}(\mathcal{M})$

References

- J. Brendle, D. A. Mejía, *Rothberger gaps in fragmented ideals*, Fund. Math. 227 (2014), no. 1, 35–68.
- L. Bukovský, I. Recław, M. Repický, *Spaces not distinguishing pointwise and quasinormal convergence of real functions*, Topology Appl. 41 (1991), no. 1-2, 25–40.
- B. Farkas, L. Soukup, *More on cardinal invariants of analytic P-ideals*, Comment. Math. Univ. Carolin. 50 (2009), no. 2, 281–295.
- R. Filipów, A. Kwela, *Yet another ideal version of the bounding number*, J. Symb. Log. 87 (2022), no. 3, 1065–1092
- R. Filipów, A. Kwela, *Two $\mathfrak{b}$ or not two $\mathfrak{b}$?* arXiv:2507.03734v1
- R. Filipów, M. Staniszewski, *Pointwise versus equal (quasi-normal) convergence via ideals*, J. Math. Anal. Appl. 422 (2015), no. 2, 995–1006.
- A. Kwela, *More on yet another ideal version of the bounding number*, to appear in Journal of Symbolic Logic
- M. Repický, *Spaces not distinguishing ideal convergences of real-valued functions*, Real Anal. Exchange 46 (2021), no. 2, 367–394.