Critical ideals for countable compact spaces

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Joint work with R. Filipów and A. Kwela.

Ideals

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is an ideal on X if

1. $X \notin \mathcal{I}$,
2. $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
3. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
4. $\mathcal{I}$ contains all finite subsets of X.

Ideals

Examples

1. $\mathrm{Fin}(X) = \{A \subseteq X : A \text{ is finite}\}$
2. $\mathrm{Fin} = \mathrm{Fin}(\omega)$
3. $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty\right\}$
4. $\mathcal{I}_d = \left\{A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap \{1,\ldots,n\}|}{n} = 0\right\}$
5. $\mathrm{Fin} \otimes \mathrm{Fin}$ is the ideal on $\omega \times \omega$ defined by

$$A \in \mathrm{Fin} \otimes \mathrm{Fin} \iff \exists i_0 \in \omega\, \forall i \geq i_0\, (|\{j \in \omega : (i,j) \in A\}| < \omega).$$

FinBW

Theorem (Bolzano-Weierstrass)

For every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is an $A \notin \mathrm{Fin}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

Definition (Kwela, 2023)

$\mathrm{FinBW}(\mathcal{I})$ is the class of all topological spaces X such that for every sequence $(x_n)_{n\in\omega}$ in X there exists $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent in X.

$X \in FinBW(\mathcal{I}) \implies X$ is sequentially compact
$X \in FinBW(\mathcal{I}) \not\Longleftarrow X$ is sequentially compact

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FinBW and Katětov order

Katětov order (Katětov, 1968)

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y respectively.
$\mathcal{J} \leq_K \mathcal{I} \iff$ there exists a function $f : X \to Y$ such that

$$A \in \mathcal{J} \implies f^{-1}[A] \in \mathcal{I}.$$

$\mathcal{I} \approx \mathcal{J} \iff$ there exists a bijection $f : X \to Y$ such that

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Theorem (Kwela, 2023)

1. $\mathcal{I} \leq_K \mathrm{Fin} \iff \mathrm{FinBW}(\mathcal{I})$ coincides with the class of all sequentially compact spaces
2. $\mathrm{Fin} \otimes \mathrm{Fin} \leq_K \mathcal{I} \iff \mathrm{FinBW}(\mathcal{I})$ coincides with the class of all finite spaces.

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Critical Ideals

Definition

Let X be a sequentially compact space. Let $D \subseteq X$ be a countable infinite subset of X. We define

$$\mathrm{conv}(D) = \{A \subseteq D : A^d \text{ is finite}\}.$$

where $A^d = \{p \in X : p \in \mathrm{cl}(A \setminus \{p\})\}$

Critical Ideals

Definition

1. If $X = [0,1]$ with the Euclidean topology, we define the ideal

$$\mathrm{conv} = \mathrm{conv}(\mathbb{Q} \cap [0,1]).$$

2. For a countable ordinal $\alpha \geq 2$ and the space $X = \omega^\alpha + 1$ with the order topology, we define the ideal

$$\mathrm{conv}_\alpha = \mathrm{conv}(\omega^\alpha + 1).$$

Proposition

If $\alpha < \beta$, then $\mathrm{conv}_\beta \leq_K \mathrm{conv}_\alpha$ and $\mathrm{conv}_\alpha \not\leq_K \mathrm{conv}_\beta$.

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$\mathrm{conv} \leq_K \mathrm{conv}_\alpha$ and $\mathrm{conv}_\alpha \not\leq_K \mathrm{conv}$ for every α.

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Combinatorial characterization

Proposition

Let $A \subseteq \omega^\alpha + 1$. T.F.A.E.

1. $A \in \mathrm{conv}_\alpha$.
2. For every increasing sequence $(\lambda_n)_{n<\omega}$ in ω^α, the set $A \cap [\lambda_n, \lambda_{n+1}]$ is finite for all but finitely many n.

Corollary

The ideals conv_2 and $\mathrm{Fin} \otimes \mathrm{Fin}$ are isomorphic.

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Countable Compact Spaces

Theorem (Mazurkiewicz-Sierpiński)

A countable space X is compact iff X is homeomorphic to the space $\omega^\alpha \cdot n + 1$ with the order topology for some countable ordinal α and $n \in \omega$.

Definition

Let $\mathbb{K}$ denote the class of all countable compact spaces and $\mathbb{K}_\alpha$ denote the class of all compact spaces which are homeomorphic to $\omega^\beta \cdot n + 1$ for some $\beta \leq \alpha$ and $n \in \omega$.
By Mazurkiewicz-Sierpiński theorem:

$$\mathbb{K} = \bigcup_{\alpha < \omega_1} \mathbb{K}_\alpha.$$

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Critical Ideals for Countable Compact Spaces

Theorem

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1. $\mathrm{conv}_{(1+\alpha)+1} \leq_K \mathcal{I}$ and $\mathrm{conv}_{1+\alpha} \not\leq_K \mathcal{I}$.
2. $\mathrm{FinBW}(\mathcal{I}) \cap \mathbb{K} = \mathbb{K}_\alpha$.

Theorem

$$\mathrm{conv}_{1+\alpha} \leq_K \mathcal{I} \iff \omega^\alpha + 1 \notin \mathrm{FinBW}(\mathcal{I}).$$

Theorem (Meza-Alcántara, 2009)

T.F.A.E.

1. $\mathrm{conv} \not\leq_K \mathcal{I}$.
2. $\mathrm{FinBW}(\mathcal{I})$ coincides with the class of all compact metric spaces in the realm of metric spaces.

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Structure

Proposition

The ideal conv_α is not the greatest lower bound of $\{\mathrm{conv}_\beta : \beta < \alpha\}$.

Definition

For every limit ordinal α define:

$$\mathrm{conv}_{<\alpha} = \{A \subseteq \omega^\alpha + 1 : \forall \beta < \alpha \; (A \cap \omega^\beta \in \mathrm{conv}_\beta)\}$$

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2. $\mathrm{conv}_\alpha \leq_K \mathrm{conv}_{<\alpha}$ and $\mathrm{conv}_{<\alpha} \nleq_K \mathrm{conv}_\alpha$.

Question

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Is there a greatest lower bound for the ideals conv_α for all α?

Topological complexity

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By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the topology of the Cantor space $\{0,1\}^\omega$ and therefore we can assign topological complexity to ideals on ω.

Proposition

1. The ideals $\mathrm{conv}_{<\alpha}$ are Π^0_5-complete ideals.
2. The ideals conv_α are Σ^0_4-complete ideals.

Thank You!

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