

Critical ideals for countable compact spaces

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Joint work with R. Filipów and A. Kwela.

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is an **ideal** on X if

- 1 $X \notin \mathcal{I}$,
- 2 $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
- 3 $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
- 4 \mathcal{I} contains all finite subsets of X .

Examples

① $\text{Fin}(X) = \{A \subseteq X : A \text{ is finite}\}$

② $\text{Fin} = \text{Fin}(\omega)$

③ $\mathcal{I}_{1/n} = \left\{ A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty \right\}$

④ $\mathcal{I}_d = \left\{ A \subseteq \omega : \lim_{n \rightarrow \infty} \frac{|A \cap \{1, \dots, n\}|}{n} = 0 \right\}$

⑤ $\text{Fin} \otimes \text{Fin}$ is the ideal on $\omega \times \omega$ defined by

$$A \in \text{Fin} \otimes \text{Fin} \iff \exists i_0 \in \omega \forall i \geq i_0 (|\{j \in \omega : (i, j) \in A\}| < \omega).$$

Theorem (Bolzano-Weierstrass)

For every sequence $(x_n)_{n \in \omega}$ in $[0, 1]$ there is an $A \notin \text{Fin}$ such that the subsequence $(x_n)_{n \in A}$ is convergent.

Definition (Kwela, 2023)

$\text{FinBW}(\mathcal{I})$ is the class of all topological spaces X such that for every sequence $(x_n)_{n \in \omega}$ in X there exists $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n \in A}$ is convergent in X .

$X \in \text{FinBW}(\mathcal{I}) \implies X$ is sequentially compact

$X \in \text{FinBW}(\mathcal{I}) \not\Leftarrow X$ is sequentially compact

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Katětov order (Katětov, 1968)

Let \mathcal{I} and \mathcal{J} be ideals on X and Y respectively.

$\mathcal{J} \leq_K \mathcal{I} \iff$ there exists a function $f : X \rightarrow Y$ such that

$$A \in \mathcal{J} \implies f^{-1}[A] \in \mathcal{I}.$$

$\mathcal{I} \approx \mathcal{J} \iff$ there exists a bijection $f : X \rightarrow Y$ such that

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Theorem (Kwela, 2023)

- $\mathcal{I} \leq_K \text{Fin} \iff \text{FinBW}(\mathcal{I})$ coincides with the class of all sequentially compact spaces
- $\text{Fin} \otimes \text{Fin} \leq_K \mathcal{I} \iff \text{FinBW}(\mathcal{I})$ coincides with the class of all finite spaces.

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Definition

Let X be a sequentially compact space. Let $D \subseteq X$ be a countable infinite subset of X . We define

$$\text{conv}(D) = \{A \subseteq D : A^d \text{ is finite}\}.$$

where $A^d = \{p \in X : p \in \text{cl}(A \setminus \{p\})\}$

Definition

- 1 If $X = [0, 1]$ with the Euclidean topology, we define the ideal

$$\text{conv} = \text{conv}(\mathbb{Q} \cap [0, 1]).$$

- 2 For a countable ordinal $\alpha \geq 2$ and the space $X = \omega^\alpha + 1$ with the order topology, we define the ideal

$$\text{conv}_\alpha = \text{conv}(\omega^\alpha + 1).$$

Proposition

If $\alpha < \beta$, then $\text{conv}_\beta \leq_K \text{conv}_\alpha$ and $\text{conv}_\alpha \not\leq_K \text{conv}_\beta$.

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Proposition

Let $A \subseteq \omega^\alpha + 1$. T.F.A.E.

- 1 $A \in \text{conv}_\alpha$.
- 2 For every increasing sequence $(\lambda_n)_{n < \omega}$ in ω^α , the set $A \cap [\lambda_n, \lambda_{n+1}]$ is finite for all but finitely many n .

Corollary

The ideals conv_2 and $\text{Fin} \otimes \text{Fin}$ are isomorphic.

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Countable Compact Spaces

Theorem (Mazurkiewicz-Sierpiński)

A countable space X is compact iff X is homeomorphic to the space $\omega^\alpha \cdot n + 1$ with the order topology for some countable ordinal α and $n \in \omega$.

Definition

Let \mathbb{K} denote the class of all countable compact spaces and \mathbb{K}_α denote the class of all compact spaces which are homeomorphic to $\omega^\beta \cdot n + 1$ for some $\beta \leq \alpha$ and $n \in \omega$.

By Mazurkiewicz-Sierpiński theorem:

$$\mathbb{K} = \bigcup_{\alpha < \omega_1} \mathbb{K}_\alpha.$$

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Critical Ideals for Countable Compact Spaces

Theorem

T.F.A.E

- 1 $\text{conv}_{(1+\alpha)+1} \leq_K \mathcal{I}$ and $\text{conv}_{1+\alpha} \not\leq_K \mathcal{I}$.
- 2 $\text{FinBW}(\mathcal{I}) \cap \mathbb{K} = \mathbb{K}_\alpha$.

Theorem

$$\text{conv}_{1+\alpha} \leq_K \mathcal{I} \iff \omega^\alpha + 1 \notin \text{FinBW}(\mathcal{I}).$$

Theorem (Meza-Alcántara, 2009)

T.F.A.E.

- 1 $\text{conv} \not\leq_K \mathcal{I}$.
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Proposition

The ideal conv_α is not the greatest lower bound of $\{\text{conv}_\beta : \beta < \alpha\}$.

Definition

For every limit ordinal α define:

$$\text{conv}_{<\alpha} = \{A \subseteq \omega^\alpha + 1 : \forall \beta < \alpha (A \cap \omega^\beta \in \text{conv}_\beta)\}$$

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Question

Is there a greatest lower bound for the ideals conv_α for all α ?

Topological complexity

By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the topology of the Cantor space $\{0, 1\}^\omega$ and therefore we can assign topological complexity to ideals on ω .

Proposition

- 1 The ideals $\text{conv}_{<\alpha}$ are Π_5^0 -complete ideals.
- 2 The ideals conv_α are Σ_4^0 -complete ideals.

Thank You!

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