Cichoń's minimum with F_σ measure zero ideal

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joint work with Diego Mejía and Ryoichi Sato

What are cardinal invariants?

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- $\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega$ and there is no single $y \in \omega^\omega$ such that every $x \in F$ is dominated by $y\}$.
- $\mathfrak{d} = \min\{|F| : F \subseteq \omega^\omega$ and for all $y \in \omega^\omega$ there is $x \in F$ such that x dominates $y\}$.

What are cardinal invariants?

(This manga is created by ChatGPT.)

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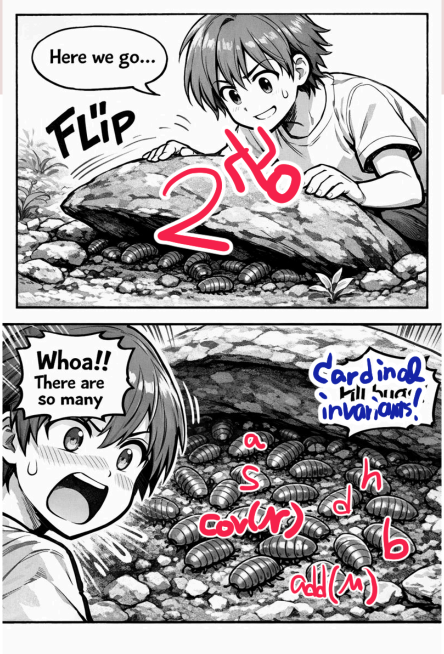

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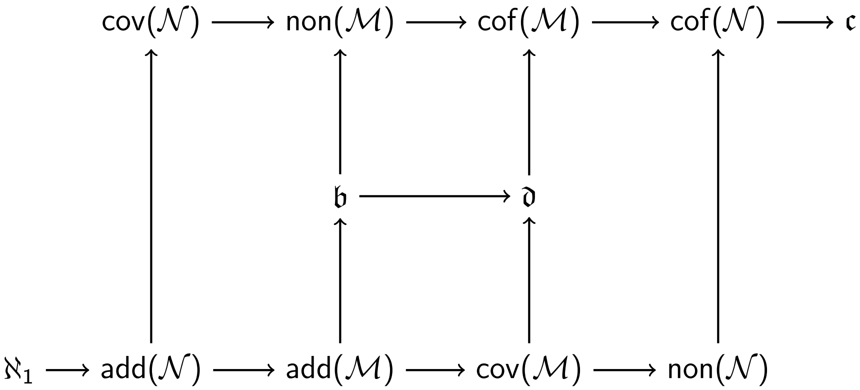
Cichoń's diagram

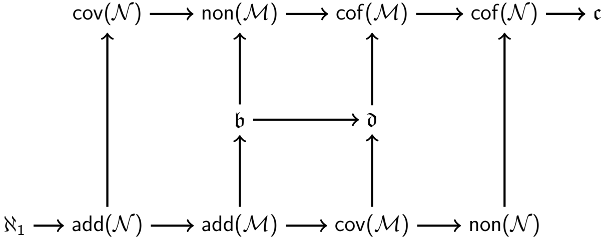
$\mathcal{N}$ and $\mathcal{M}$ denote the null ideal and the meager ideal, respectively.
$x \to y$ means ZFC proves $x \leqslant y$.



Cichoń's "minimum"

There are 23 many assignments of $\aleph_1$ and $\aleph_2$ to the cardinal invariants appearing in Cichoń's diagram not violating Cichoń's diagram and the constraints $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

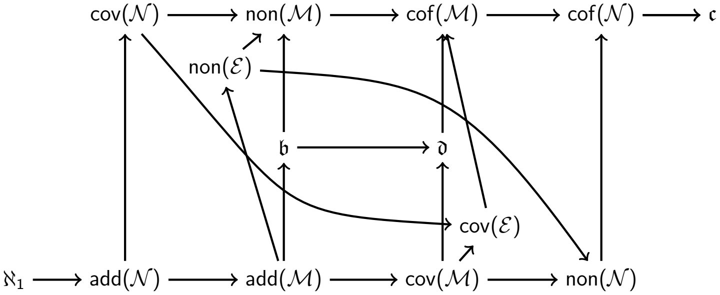
Each of them are forceable (mainly due to Bartoszyński–Judah–Shelah). We here call it Cichoń's "minimum".



The ideal $\mathcal{E}$

Let $\mathcal{E}$ be the σ-ideal generated by F_σ null sets.

Cichoń's diagram with $\mathcal{E}$



It is known that $\mathrm{add}(\mathcal{E}) = \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{E}) = \mathrm{cof}(\mathcal{M})$.
Also $\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} \leqslant \mathrm{cov}(\mathcal{E}) \leqslant \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$ and $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} \leqslant \mathrm{non}(\mathcal{E}) \leqslant \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$.

Prior studies

- Research on cardinal invariants of $\mathcal{E}$ dates back to Bartoszynski–Shelah 1992.
- Cichoń's "minimum" completed by Bartoszynski–Judah–Shelah 1993.
- Cichoń's maximum was achieved in 2019, and Cichoń's maximum without large cardinals in 2022.
- Cichoń's maximum with $\mathcal{E}$ was achieved by Yamazoe in 2026.

Note that Cichoń's maximum with $\mathcal{E}$ does not imply Cichoń's minimum with $\mathcal{E}$, because the order of cardinal invariants is fixed in each Cichoń's maximum model.

Cichoń's minimum with $\mathcal{E}$

There are 36 many assignments of $\aleph_1$ and $\aleph_2$ to the cardinal invariants appearing in Cichoń's diagram and $\mathrm{cov}(\mathcal{E})$ and $\mathrm{non}(\mathcal{E})$ not violating currently known ZFC results.

Our conjeture is all of them are forceable.

Models for $\mathrm{non}(\mathcal{M}) \leqslant \mathrm{cov}(\mathcal{M})$

There are 16 constellations in which $\mathrm{non}(\mathcal{M}) \leqslant \mathrm{cov}(\mathcal{M})$ holds.

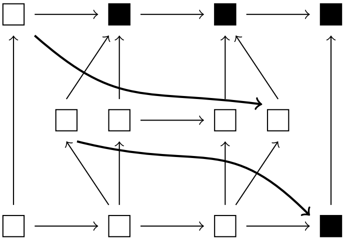
One of these constellations is CH.

Other 15 constellations can be obtained by finite support iteration, using Cohen forcing, random forcing, Hechler forcing, amoeba forcing, eventually different forcing and Kellner–Shelah–Tănasie forcing $\tilde{\mathbb{E}}$.

Models for $\mathrm{cov}(\mathcal{M}) < \mathrm{non}(\mathcal{M})$

These cases, we cannot use finite support iteration. So we use countable support iteration.

A model where non($\mathcal{M}$) and non($\mathcal{N}$) are large



This constellation is achieved by the countable support iteration of length ω_2, interweaving $\mathbf{PT}_{f,g}$ and $\mathbf{S}_{g,g*}$. $\mathbf{PT}_{f,g}$ increases non($\mathcal{M}$) and $\mathbf{S}_{g,g*}$ increases non($\mathcal{N}$). It has been known that $\mathfrak{d}$ and cov($\mathcal{N}$) are small in this model (thus cov($\mathcal{E}$) is also small).

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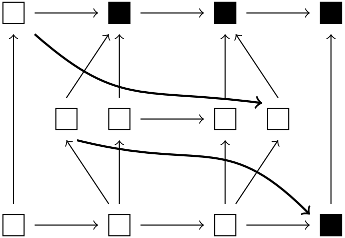


New ingredient 1: Every limsup forcing (including $\mathbf{S}_{g,g*}$) preserves all relations for the left side of Cichoń's diagram.
New ingredient 2: $\mathbf{PT}_{f,g}$ preserves the relation for non($\mathcal{E}$).

Ingredients

Ingredients we have done are:

- Every limsup forcing (including $\mathbf{S}_{g,g*}$) preserves all relations for the left side of Cichoń's diagram.
- $\mathbf{PT}_{f,g}$ preserves the relation for $\mathrm{non}(\mathcal{E})$.
- Every proper forcing having a suitable fragment of Laver property preserves the relation for $\mathrm{cov}(\mathcal{N})$.

Ingredients we have not done are:

- $P_{\mathcal{E}}$ has the property preserving $\mathrm{non}(\mathcal{N})$.
- $P_{\mathcal{E}}$ has the property preserving $\mathrm{cov}(\mathcal{M})$.
- Finding alternative forcing notions to $\mathbf{PT}_{f,g}$ and $\mathbf{Q}$ (a creature forcing increasing $\mathrm{non}(\mathcal{E})$).

Table (1/2)

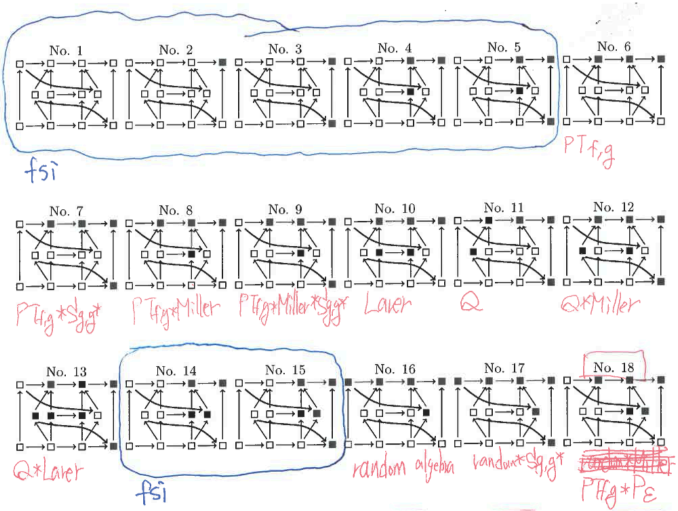
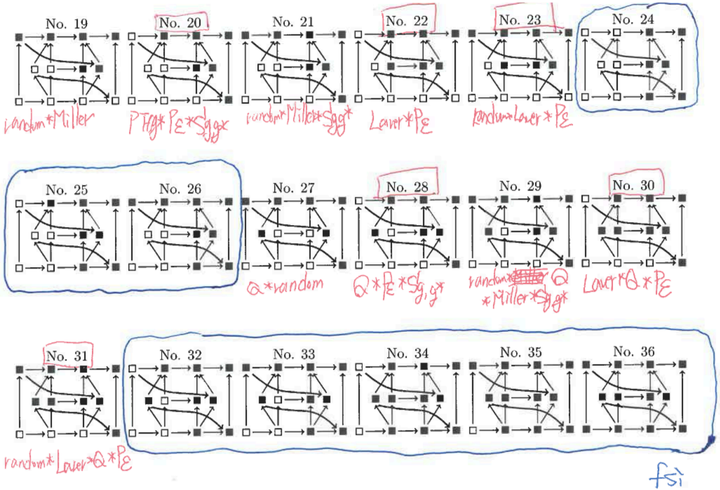

No. 1
No. 2
No. 3
No. 4
No. 5
No. 6
fsi
PTf,g
No. 7
No. 8
No. 9
No. 10
No. 11
No. 12
PTf,g*Sg,g*
PTf,g*Miller
PTf,g*Miller*Sg,g*
Laver
Q
Q*Miller
No. 13
No. 14
No. 15
No. 16
No. 17
No. 18
Q*Laver
fsi
random algebra
random*Sg,g*
PTf,g*PE

Table (2/2)


No. 19
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No. 26
No. 27
No. 28
No. 29
No. 30
No. 31
No. 32
No. 33
No. 34
No. 35
No. 36

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