On minimal dimension of normed barrelled spaces

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Motivation — Three Fundamental Theorems

Uniform Boundedness Principle

X, Y – Banach spaces
$\mathcal{F}$ – family of continuous operators $X \to Y$

If $\sup_{T\in\mathcal{F}} \|T(x)\| < \infty$ for every $x \in X$, then $\sup_{T\in\mathcal{F}} \|T\| < \infty$.

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If T has closed graph, then T is continuous.

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Open Mapping Theorem

X, Y – Banach spaces
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If T is onto, then T is open.

Uniform Boundedness Principle according to Rudin's *F.A.*

Uniform Boundedness Principle (General Form)

X – F-space (i.e. completely invariant-metrizable topological vector space)
Y – topological vector space
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X – F-space (i.e. completely invariant-metrizable topological vector space)
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If $\{T(x)\colon T \in \mathcal{F}\}$ is bounded in Y for each $x \in X$, then $\mathcal{F}$ is equicontinuous.

Barrelled spaces

Definition

If A is a subset of a tvs X which is closed, convex, balanced, and absorbing, then A is a *barrel*.

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A tvs X is *barrelled* if every barrel in X is a neighborhood of 0.

Characterizations of barrelled spaces

Theorem

Let X be a tvs. TFAE:

- X is barrelled,

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Theorem

Let Y be a locally convex space. TFAE:

- Y is barrelled,
- for any lcs X and operator $T\colon X \to Y$, if T is onto, then for any nhbd U of 0 in X the closure $\overline{T[U]}$ is a nhbd of 0 in Y.

Examples of barrelled spaces

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- $C_p(X)$ for X infinite pseudocompact

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Theorem (folklore; probably Schachermayer)

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Theorem (Buchwalter–Schmets)

Let X be a Tychonoff space. TFAE:

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- every functionally bounded subset of X is finite.

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Corollary

$\mathfrak{c} = \min\{\mathit{dim}(X) \colon X \text{ is an infinite-dimensional Fréchet space}\}$

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Every vector space V can be endowed with a topology τ making it a barrelled space.

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So, consider the <u>vector</u> space c_{00} to get the following

Corollary

$\omega = \min\{dim(X) \colon X \text{ is an infinite-dimensional barrelled space}\}$

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What is the minimal dimension $\mathfrak{nb}$ of an infinite-dimensional normed barrelled space?

Corollary

$\mathfrak{b} \leqslant \mathfrak{nb} \leqslant \mathfrak{c}$

Dimension of normed barrelled spaces

Theorem (S.)

If $\mathrm{cof}(\mathcal{N}) = \omega_n$ for some $n \in \omega$, then there is a normed barrelled space of dimension $\mathrm{cof}(\mathcal{N})$.

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Theorem (S.–Zdomskyy)

Let $\mathbb{P} \in \{\text{Sacks, Silver, Miller}\}$. Then, in any $\mathbb{P}$-generic extension $V[G]$, the linear space spanned by clopen subsets of the Stone space $St(\wp(\omega)^V)$ of the ground model Boolean algebra $\wp(\omega)^V$ endowed with the supremum norm is barrelled and of dimension ω_1.

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If $\mathbb{P}$ adds a Cohen real, a random real, or a dominating real (e.g. $\mathbb{P} \in \{$Cohen, random, Hechler, Mathias, Laver$\}$), then the above theorem does not hold for $\mathbb{P}$.

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So, almost $\mathrm{cov}(\mathcal{N}) \leqslant \mathfrak{nb}$...

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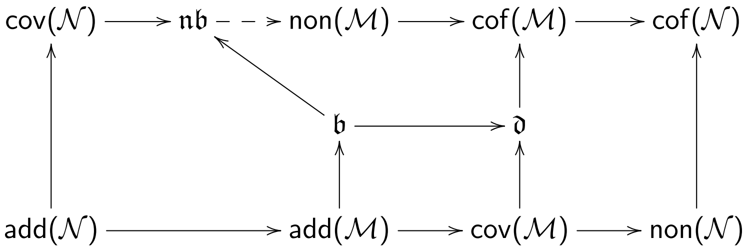
$\max(\mathfrak{b}, \mathrm{cov}(\mathcal{N})) \leqslant \mathfrak{nb} \leqslant \mathrm{non}(\mathcal{M})$

Considering any model with $\mathfrak{b} < \mathrm{cov}(\mathcal{N})$, we get the following

Corollary

Consistently, there exists an infinite-dimensional <u>metrizable</u> barrelled space X such that for each infinite-dimensional <u>normed</u> barrelled space Y we have $dim(X) < dim(Y)$.

Cichoń's diagram and $\mathfrak{nb}$



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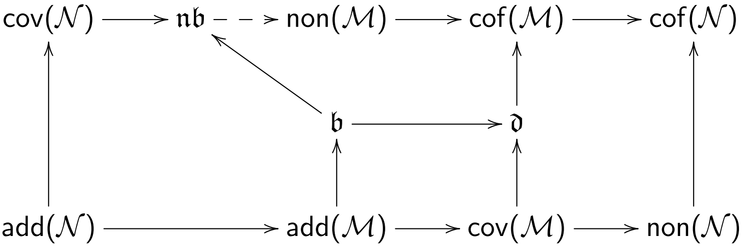


Question

$\mathfrak{nb} = \mathrm{non}(\mathcal{M})$?

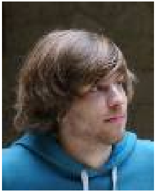
The end

Thank you for the attention!

Thank you, Guys!!!

Adam Bartoš

David Chodounský



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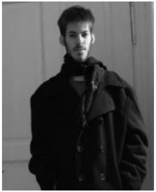

Chris Lambie-Hanson