Rudin–Blass Ordering of Measures

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Measures and Filters

Measure $\equiv$ finitely additive probability measure. Unless otherwise specified, also vanishing on points.

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$$d_{\mathcal{U}}(A) = \mathcal{U}\text{-}\lim_n \frac{|A \cap n|}{n}.$$

Rudin–Blass ordering

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For Rudin–Keisler, drop the finite-to-one requirement.

Q-measures

Selectors

Given $\langle P_n \rangle$ – a partition of ω, a *selector* is a set $S \subseteq \omega$ such that for each n

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Q-points and Q-measures

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$S \in \mathcal{U}$	$\mu(S) > 0$	$\mu(S) = 1.$

Q vs RB

Fact

Q-points are exactly the Rudin–Blass minimal ultrafilters.

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Fact

Q-measures are Rudin–Blass minimal.

Theorem

Consistently, there exists a Rudin-Blass minimal measure, which is not Q^+.

On existence

Fact

Under *Filter Dichotomy* there is an ultrafilter RB-below any measure and there are no Q-points.

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There exists a Q-point whenever there exists a Q-measure.

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Under *Filter Dichotomy* there is an ultrafilter RB-below any measure and there are no Q-points. So there are no Q^+-measures and no RB-minimal measures.

Fact

There exists a Q-point whenever there exists a Q-measure.

$d_{\mathcal{U}}$ is never Q^+.

Atomless Q-measures

Theorem (Avilés, Martinez-Cervantes, Poveda, Sáenz)

Under $\mathfrak{d} = \mathrm{cov}(\mathcal{M})$, every atomless measure defined on a Boolean algebra of size $< \mathfrak{d}$ can be extended to an atomless Q-measure.

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If there exists an atomless Q-measure, then there are $2^{\mathfrak{c}}$ different Q-points.

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If there are infinitely many pairwise RK-incompatible selective ultrafilters, then there is an atomless Q-measure.

Selective measures and Rudin–Keisler ordering

Selective measures

A measure μ is selective if for any partition $\langle P_n \rangle$ of ω there is a selector S with

$$\mu(S) = 1 - \sum_n \mu(P_n).$$

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Measure is selective if and only if it is a P-measure and a Q-measure.

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Proposition

Selective measures are Rudin–Keisler minimal.

Construction Framework (Mokobodzki) skip

We are working with sequences of measures with finite support (there is a finite set of full measure, *so not vanishing on points*). Focus on limit behavior.

Consider δ given by $\delta_n(A) = \begin{cases} 1, & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}$

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$a \preceq b$ if we have $a_n \in \mathrm{conv}\{b_m : m \geqslant n\}$ for all n.

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$a \preceq^* b$ if we have $a_n \in \mathrm{conv}\{b_m : m \geqslant n\}$ for all but finitely many n.

Construction Framework (Mokobodzki)

$a \preceq^{*} b$ if $a_n \in \mathrm{conv}\{b_m : m \geqslant n\}$ for almost all n.

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Let $\lambda_a(X) = \lim_n a_n(X)$ whenever possible.

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Proposition

For any $a \preceq \delta$ and f finite-to-one there are $b \preceq a$ and g fin-to-one such, that

$$\forall_X \forall_n \quad \left| b_n(X) - b_n\left((gf)^{-1}[X]\right) \right| < 2^{-n}$$

Construction Scheme

- Assume CH
- Enumerate $[\omega]^\omega$ as $\langle A_\alpha : \alpha < \omega_1 \rangle$ and finite to one functions as $\langle f_\alpha : \alpha < \omega_1 \rangle$.
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Let $\lambda(X) = \lambda_{a^\alpha}(X) = \lim_n a^\alpha_n(X)$ for any α where the limit exists.

Improvements

Restrict ourselves to $\mathbb{Q}$-convex combinations. Consider

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This ordering is $<\mathfrak{p}$-closed.

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Theorem

Under $\mathfrak{p} = \mathfrak{c}$ there is a Rudin–Blass minimal measure, which is not Q^+. It can be made a P-measure, and so Rudin–Keisler minimal.

Concluding Remarks

Theorem

If μ is Rudin–Blass minimal, then $\mu = \mu_0 + \sum_n \nu_n$ where μ_0 is a Q-measure and each ν_n is not Q^+.

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Questions

Can all/none RB-minimal measures be Q?
Are all Rudin–Keisler minimal measures P-measures?
Does existence of Q^+ measures imply the existence of Q-measures?