Changing the Length of the Borel Hierarchy

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Borel sets

Definition

Let X be a topological space. Then the Borel hierarchy on X is stratified into classes $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$, where $\mathbf{\Sigma}^0_1(X)$ is the class of open sets,

$$\mathbf{\Sigma}^0_\alpha(X) = \left\{ \bigcup_{i<\omega} A_i \mid \langle A_i | i < \omega \rangle \subseteq \bigcup_{\beta<\alpha} \mathbf{\Pi}^0_\beta(X) \right\}$$

and $\mathbf{\Pi}^0_\alpha(X) = \{X \setminus A \mid A \in \mathbf{\Sigma}^0_\alpha(X)\}$.

Order

Definition

The order (or length) of the Borel hierarchy on X is defined as

$$\mathrm{ord}(X) = \min\left\{\alpha : \mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Pi}^0_\alpha(X) = \mathbf{Bor}(X)\right\}.$$

Observations

Fix $X \subseteq {}^{\omega}\omega$.

- $1 \leq \text{ord}(X) \leq \omega_1$.
- If X is countable, then $\text{ord}(X) \leq 2$. Furthermore, $\text{ord}(X) = 1$ if and only if X is discrete, which is an absolute property.
- $\text{ord}({}^{\omega}2) = \omega_1$.
- If $X \hookrightarrow Y$, then $\text{ord}(X) \leq \text{ord}(Y)$. In particular, every space with a perfect subset has order ω_1.
- If X is a Luzin set, then $\text{ord}(X) = 3$. (Poprougenko 1930)

Guiding Question

What can $\text{ord}(X)$ be?

Coding Borel sets

Definition

Let a well-founded tree $T \subseteq {}^{<\omega}\omega$ and a function $f : \mathrm{leaf}(T) \to {}^{<\omega}\omega$ be given. For any node $\eta \in T$, define the interpretation of η as

$$\mathcal{I}_T^f(\eta) = \begin{cases} [f(\eta)] \cap X & \text{if } \eta \in \mathrm{leaf}(T) \\ \bigcap_{\nu \in \mathrm{succ}(\eta)} X \setminus \mathcal{I}_T^f(\nu) & \text{otherwise.} \end{cases}$$

If $\mathrm{rk}_T(\eta) = \alpha$, then $\mathcal{I}_T^f(\eta) \in \mathbf{\Pi}_\alpha^0(X)$. Furthermore, $B \in \mathbf{\Pi}_\alpha^0(X)$ if and only if $B = \mathcal{I}_T^f(\eta)$ for some tree T with $\mathrm{rk}(T) = \alpha$ and $f : \mathrm{leaf}(T) \to {}^{<\omega}\omega$. The pair $\langle T, f\rangle$ is a **Borel code** for B. Note that for every $\alpha < \omega_1$ there exists a *universal* such tree $T = T_\alpha$ as above.

α-forcing

Suppose $X \subseteq {}^{\omega}\omega$, $A, B \subseteq X$ are disjoint and $1 < \alpha < \omega_1$.
The partial order $\mathbb{BM}_\alpha(A, B, X)$ of α-forcing (Miller 1979) consists of pairs $p = \langle f_p, R_p \rangle$, where

- f_p is a finite partial function assigning basic clopen sets to leaves of a (canonically chosen) tree T_α of rank α;
- R_p is a finite relation between inner nodes of the tree and elements of the space X;
- the function f_p and R_p *cohere* to generically define a $\mathbf{\Pi}^0_\alpha(X)$ code for a subset of X;
- the parameters A, B restrict the form of that subset.

Properties of α-forcing

A $\mathbb{BM}_\alpha(A, B, X)$-generic G induces a function

$$f_G : \mathsf{leaf}(T_\alpha) \to {}^{<\omega}\omega,$$

hence for every $\eta \in T_\alpha$ a $G_\eta \in \mathbf{\Pi}^0_{\mathsf{rk}_{T_\alpha}(\eta)}(X)$. A pair $\langle \eta, x \rangle \in R_p$ means the promise

$$p \Vdash x \in G_\eta.$$

Properties of α-forcing

Lemma

For $\eta \in \mathrm{nonleaf}(T_\alpha)$ *and* $x \in X$ *we have*

$$(\exists p \in G : \langle \eta, x \rangle \in R_p) \Leftrightarrow x \in G_\eta.$$

α-forcing

Lemma

If G is $\mathbb{BM}_\alpha(A, B, X)$-generic, then $A \subseteq G_\emptyset \subseteq X \setminus B$.

Corollary

$\mathbb{BM}_\alpha(A, X \setminus A, X) \Vdash A \in \mathbf{\Pi}^0_\alpha(X)$.

Two forcings of interest: $\mathbb{BM}_\alpha(\emptyset, \emptyset, X)$ and $\mathbb{BM}_\alpha(A, X \setminus A, X)$.

A model

Theorem (Miller 1979)

Let $X \subseteq {}^{\omega}\omega$ be uncountable and $1 < \alpha < \omega_1$ a successor ordinal. Then there is a c.c.c. forcing extension with $V[G] \models \mathrm{ord}(X) = \alpha$.

Proof Strategy.

Finite support iteration of α-forcing, where we iterate sufficiently long to catch all Borel subsets of X using a bookkeeping argument as parameters for a $\mathbb{BM}_\alpha(A, X \setminus A, X)$-generic. □

After this forcing, we have $V[G] \models \mathrm{ord}(X) \leq \alpha$.

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How can we ensure $V[G] \models \mathrm{ord}(X) \geq \alpha$?

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First add a $\mathbb{BM}_{\alpha-1}(\emptyset, \emptyset, X)$-generic.
Use ranked forcing techniques to ensure

$$V[G] \models G_{\emptyset}(0) \notin \mathbf{\Sigma}^0_{\alpha-1}(X)!$$

Ranked forcing

Definition (Miller)

Let $\mathbb{P}$ be a forcing notion. We say $\text{rk} : \mathbb{P} \to \text{Ord} \cup \{\infty\}$ is a *rank function* on $\mathbb{P}$ if the following holds:
For each $p \in \mathbb{P}$ and ordinal β there exists a $q \in \mathbb{P}$ with $q \mid\mid p$ and $\text{rk}(q) \leq \beta$ such that for each $r \in \mathbb{P}, \text{rk}(r) < \beta$ we have

$$r \mid\mid q \implies r \mid\mid p.$$

The forcing $\mathbb{BM}_\alpha(A, B, X)$ admits a **sufficiently rich family of rank functions**.

Preserving the generic Π-set

The following theorem is formulated in [Ch] to encompass several arguments of Miller.

Theorem

Suppose $1 < \alpha < \omega_1$, $X \subseteq {}^{\omega}2$ *and* $\underset{\sim}{\mathcal{P}}$ *is a* $\mathbb{BM}_\alpha(\emptyset, \emptyset, X)$*-name for a c.c.c. forcing notion. Write* $\mathbb{P} = \mathbb{BM}_\alpha(\emptyset, \emptyset, X) \star \underset{\sim}{\mathcal{P}}$*. Assume that there exists a* **sufficiently rich family of rank functions** *on* $\mathbb{P}$*. Then for each uncountable* $Y \subseteq X$

$$\mathbb{P} \Vdash G_\emptyset(0) \cap Y \notin \mathbf{\Sigma}^0_\alpha(Y).$$

Iterating

From now on, we consider finite support iterations

$$\mathbb{P} = \langle \mathbb{P}_\gamma, \underset{\sim}{\mathbb{Q}}_\gamma : \gamma < \gamma^* \rangle,$$

where $\mathbb{P}_\gamma \Vdash \underset{\sim}{\mathbb{Q}}_\gamma = \mathbb{BM}_{\alpha_\gamma}(\underset{\sim}{A}_\gamma, \underset{\sim}{B}_\gamma, \underset{\sim}{X}_\gamma)$. Without loss of generality $A_0 = B_0 = \emptyset$.

Most iterations cannot be ranked

Example

Consider the two-step iteration

$$\mathbb{P} = \mathbb{BM}_\alpha(\emptyset, \emptyset, X) \star \mathbb{BM}_\alpha(X \setminus G_\emptyset(0), G_\emptyset(0), X).$$

Then $\mathbb{P} \Vdash G_\emptyset(0) \in \mathbf{\Sigma}^0_\alpha(X)$, so by the previous theorem $\mathbb{P}$ cannot be ranked!

Two results

Miller's main two results concerning α-forcing are the following:

Iteration Ranked (Miller 1979, Theorem 34)

$\underset{\sim}{X}_\gamma = X \in V, \gamma < \gamma^*$ and $\alpha_\gamma = \alpha_0 + 1, 0 < \gamma < \gamma^*$.

Theorem (Miller 1979, Theorem 34)

For every uncountable $X \subseteq {}^\omega\omega$ and successor ordinal $\alpha < \omega_1$ there is a c.c.c forcing extension

$$V[G] \models \mathrm{ord}(X) = \alpha.$$

Two results

Miller's main two results concerning α-forcing are the following:

Iteration Ranked (Miller 1979, Theorem 22)

$\underset{\sim}{X}_\gamma = {}^\omega\omega \cap V^{\mathbb{P}_\gamma}$ and $\Vdash \underset{\sim}{A}_\gamma = \underset{\sim}{B}_\gamma = \emptyset$ for all $\gamma < \gamma^*$.

Theorem (Miller 1979, Theorem 22)

There is a model of

$$\forall X \subseteq {}^\omega\omega : |X| \leq \aleph_0 \vee \mathrm{ord}(X) = \omega_1.$$

One result

The two proofs are very different, but they can be unified to get the following:

Iteration Ranked (Ch.)

$\underset{\sim}{X}_\gamma = {}^\omega\omega \cap V^{\mathbb{P}_\gamma}$, either $\Vdash \underset{\sim}{A}_\gamma = \underset{\sim}{B}_\gamma = \emptyset$ or $\alpha_\gamma > \alpha_0$.

It would require a much more sophisticated analysis to drop the requirement $(\nVdash \underset{\sim}{A}_\gamma = \underset{\sim}{B}_\gamma = \emptyset) \Rightarrow \alpha_\gamma > \alpha_0$!

One result

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$\underset{\sim}{X}_\gamma = {}^\omega\omega \cap V^{\mathbb{P}_\gamma}$, either $\Vdash \underset{\sim}{A}_\gamma = \underset{\sim}{B}_\gamma = \emptyset$ or $\alpha_\gamma > \alpha_0$.

Theorem (Ch.)

Consistently $\mathrm{ord}(X) \geq \alpha$ *for all uncountable* $X \subseteq {}^\omega\omega$ *and* $\mathrm{ord}(X) = \alpha$ *for* $|X| < 2^\omega$.

Note that the perfect set property starts taking hold at $|X| = 2^\omega$.

Generalized descriptive set theory

Let $\kappa = \kappa^{<\kappa}$ be a regular cardinal.
The analogue we are now interested in are the κ-Borel subsets of $X \subseteq {}^{\kappa}\kappa$!

Definition

The higher Baire space ${}^{\kappa}\kappa$ is endowed with the *bounded* topology generated by the sets $[s] = \{x | s \subseteq x\}$ for $s \in {}^{<\kappa}\kappa$.
For $X \subseteq {}^{\kappa}\kappa$, the κ-Borel hierarchy is generated by taking $\leq\kappa$-unions and intersections.

Definition

Let $\text{ord}_\kappa(X)$ be the least ordinal α such that

$$\kappa\text{-}\mathbf{\Sigma}^0_\alpha(X) = \kappa\text{-}\mathbf{\Pi}^0_\alpha(X) = \kappa\text{-}\mathbf{Bor}(X).$$

We have $1 \leq \text{ord}_\kappa(X) \leq \kappa^+$.

Higher α-forcing

One can attempt to generalize the requisite notions and define a higher version of α-forcing. The presence of nodes $\eta \in T$ with limit rank $<\kappa$ makes this difficult, necessitating the use of a more complicated combinatorial structure. Let us write $\mathbb{BM}_\alpha^\kappa(A, B, X)$ for the κ-version of α-forcing.

Instead of finite support, we now work with $<\kappa$-supported iterations of κ^+-c.c. forcings.

Preserving the generic Π-set, revisited

Theorem

Suppose $\lambda \geq \kappa$, $1 < \alpha < \kappa^+$, $X \subseteq {}^{\kappa}2$ *and* $\underset{\sim}{\mathcal{P}}$ *is a* $\mathbb{BM}^{\kappa}_{\alpha}(\emptyset, \emptyset, X)$*-name for a* λ^+*-c.c. forcing notion. Write* $\mathbb{P} = \mathbb{BM}^{\kappa}_{\alpha}(\emptyset, \emptyset, X) \star \underset{\sim}{\mathcal{P}}$*. Lastly, assume that* $\mathbb{P}$ *admits a* **sufficiently rich family of rank functions.**

Then for each $Y \subseteq X$ *with* $|Y| > \lambda$*,*

$$\mathbb{P} \Vdash G_{\emptyset}(0) \cap Y \notin \lambda\text{-}\mathbf{\Sigma}^0_{\alpha}(Y).$$

Spaces of finite order

Iteration Ranked (Agostini, Ch., Motto Ros, Pitton)

$\underset{\sim}{X}_\gamma = X \in V$, $\alpha_\gamma = \alpha_0 + 1 < \omega$ for $\gamma > 0$.

Theorem (Agostini, Ch., Motto Ros, Pitton)

Suppose $X \subseteq {}^\kappa\kappa$ is of size $|X| > \kappa$ and $1 < \alpha < \omega$. Then there is a $<\kappa$-closed, κ^+-c.c. forcing extension such that

$$V[G] \models \text{ord}_\kappa(X) = \alpha.$$

Infinite order

Iteration Ranked (Ch.)

- $\underset{\sim}{X}_\gamma = X_\gamma \in V$,
- either $\alpha_\gamma > \alpha_0$ or $\mathbb{P}_\gamma \Vdash \underset{\sim}{A}_\gamma = \underset{\sim}{B}_\gamma = \emptyset$ or $|X_\gamma| < |X_0|$.

Theorem (Ch.)

Suppose $X \subseteq {}^\kappa\kappa, |X| > \kappa$ and $1 < \alpha < \kappa^+$ is a successor. Then there is a $<\kappa$-closed, κ^+-c.c. forcing extension such that

$$V[G] \models \mathrm{ord}_\kappa(X) = \alpha.$$

Separating by size

Theorem (Ch.)

Let f be a function assigning to each cardinal λ with $\kappa < \lambda \le 2^\kappa$ an ordinal $1 < f(\lambda) \le \kappa^+$ such that

1. *f is (not necessarily strictly) increasing,*
2. $f(2^\kappa) = \kappa^+$,
3. *if λ is a successor cardinal, then $f(\lambda)$ is a successor ordinal or $f(\lambda) = \kappa^+$,*
4. *Continuity: if λ is a limit cardinal, then $f(\lambda) = \sup_{\lambda' < \lambda} f(\lambda')$.*

Then there exists a $<\kappa$-closed, κ^+-c.c. generic extension $V[G]$ of the universe such that

$$V[G] \models \forall X \subseteq {}^\kappa\kappa, X \in V, |X| > \kappa : \mathrm{ord}_\kappa(X) = f(|X|).$$

Ranked iterations

Type \ Setting	ω	$\kappa = \kappa^{<\kappa}$, α finite	$\kappa = \kappa^{<\kappa}$, $1 < \alpha < \kappa^+$
fixed space X	Miller 1979	Agostini, Ch., Motto Ros, Pitton	Ch.
$\underset{\sim}{X}_\gamma = {}^\omega\omega \cap V^{\mathbb{P}_\gamma}$, only increase	Miller 1979	Ch.: at least for $\gamma^* < \omega$	
$\underset{\sim}{X}_\gamma = {}^\omega\omega \cap V^{\mathbb{P}_\gamma}$, one α	Ch.	Possibly at least for $\gamma^* < \omega$	
separate by size: ground model	essentially Miller '79 (Th. 34,52)	Ch.	
separate by size: all subsets	?		

Děkuji za pozornost!

Leaving ω behind and reaping the benefits

Definability of X

Suppose $X \subseteq {}^{\omega}\omega$ is itself definable (Borel). Then $\mathrm{ord}_{\omega}(X) \in \{1, 2, \omega_1\}$ by the perfect set property.
This no longer has to be the case in the generalized setting: consistently, the κ-perfect set property fails even for closed sets!

Leaving ω behind and reaping the benefits

Combining arguments of Silver, Lücke, Agostini/Ch./Motto Ros/Pitton and Ch. gives us:

Theorem

For $\kappa > \omega$ and $1 < \alpha < \kappa^+$ there consistently exists a κ-Borel set B with $\text{ord}_\kappa(B) = \alpha$.

Moreover, in the preceding theorems, whenever a set $X \in V$ is κ-Borel with no perfect subset, it will retain the **same** definition through its κ-Borel code in the forcing extension. One can arrange $X \in \kappa\text{-}\mathbf{\Sigma}^0_2$ with no perfect subset for every $X \in V$.

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