Perfect set dichotomy theorem in generalized Solovay model

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Solovay model

Definition (Solovay model)

Let κ be an inaccessible cardinal and let G be a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter over V. Then we call $V(\mathbb{R}^{V[G]}) = \mathsf{HOD}^{V[G]}_{V \cup \mathbb{R}^{V[G]}}$ the **Solovay model** for κ.

Theorem (Solovay)

Let κ be an inaccessible cardinal and let G be a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter over V. Then $V(\mathbb{R}^{V[G]})$ is a model of $\mathsf{ZF} + \mathsf{DC}$ in which all set of reals have regularity properties, such as the Baire property, Lebesgue measurability and the perfect set property.

In the Solovay model, every subset of the reals has " regularity " properties of definable sets.

Perfect set dichotomy

We focus on a dichotomy theorem for equivalence relations on the reals.

The *perfect set dichotomy* for an equivalence relation E on $\mathbb{R}$ asserts that either
(1) $\mathbb{R}/E$ is well-orderable, or
(2) there is a perfect set of pairwise E-inequivalent reals.

Perfect set dichotomy

Classical results establish the perfect set dichotomy for definable equivalence relations.

Theorem (Silver)

Let E be a $\mathbf{\Pi}^1_1$ equivalence relation on $\mathbb{R}$. Then either there is an injection from $\mathbb{R}/E$ to ω or there is a perfect set of pairwise E-inequivalent reals.

Theorem (Burgess)

Let E be a $\mathbf{\Sigma}^1_1$ equivalence relation on $\mathbb{R}$. Then either there is an injection from $\mathbb{R}/E$ to ω_1 or there is a perfect set of pairwise E-inequivalent reals.

Perfect set dichotomy

How about models in which every set of reals has regularity properties of definable sets?

Theorem (Woodin)

Assume ZF + AD + $V = L(\mathbb{R})$. Then the perfect set dichotomy holds for all equivalence relations on $\mathbb{R}$.

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Theorem (Woodin)

Assume ZF + AD + $V = L(\mathbb{R})$. Then the perfect set dichotomy holds for all equivalence relations on $\mathbb{R}$.

Theorem (Sakai, T.)

In the Solovay model $V(\mathbb{R}^{V[G]})$ for an inaccessible cardinal, the perfect set dichotomy holds for all equivalence relations on $\mathbb{R}$.

Sketch of Proof

We work in $V[G]$. Let E be an equivalence relation on $\mathbb{R}$ in $V(\mathbb{R}^{V[G]})$.

$\Omega := \{(\xi, p, \dot{x}) \mid \xi < \kappa, p \in \mathrm{Coll}(\omega, \xi), \dot{x}$ is a $\mathrm{Coll}(\omega, \xi)$-name for a real with $\dot{x} \in \mathcal{H}_\kappa^V \}$.

For $(\xi, p, \dot{x}) \in \Omega$,

$$\mathrm{Eval}(\xi, p, \dot{x}) := \{\dot{x}[h] \mid h \text{ is a } \mathrm{Coll}(\omega, \xi)\text{-generic filter over } V \text{ with } p \in h\}.$$

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- **Case I** :
  $\forall(\xi, q, \dot{x}) \in \Omega \ \exists p \leqslant q \ \ (\forall x_0, x_1 \in \mathrm{Eval}(\xi, p, \dot{x}) \ \ x_0 \ E \ x_1)$.
- **Case II** :
  $\exists(\xi, q, \dot{x}) \in \Omega \ \forall p \leqslant q \ \ (\exists x_0, x_1 \in \mathrm{Eval}(\xi, p, \dot{x}) \ x_0 \not{E} \ x_1)$.

Sketch of Proof (Case I)

Case I : $\forall(\xi, q, \dot{x}) \in \Omega\ \exists p \leqslant q\ \ (\forall x_0, x_1 \in \mathrm{Eval}(\xi, p, \dot{x})\ \ x_0\ E\ x_1)$.

We show that $\mathbb{R}/E$ is well-orderable.

Suppose $a \in \mathbb{R}$. We can take $(\xi, p, \dot{x}) \in \Omega$ such that $a \in \mathrm{Eval}(\xi, p, \dot{x})$ and $\forall x_0, x_1 \in \mathrm{Eval}(\xi, p, \dot{x})\ \ x_0\ E\ x_1$.
Thus for all $A \in \mathbb{R}/E$ there is $(\xi_A, p_A, \dot{x}_A) \in \Omega$ such that

$$\mathrm{Eval}(\xi_A, p_A, \dot{x}_A) \subseteq A.$$

Since AC holds in V and $V \subseteq V(\mathbb{R}^{V[G]})$, we can take a well-ordering $\leqslant_\Omega$ of Ω.
Define a well-order $\preceq_E$ on $\mathbb{R}/E$ by $A \preceq_E B :\Leftrightarrow \mathrm{Eval}(\xi_A, p_A, \dot{x}_A) \leqslant_\Omega \mathrm{Eval}(\xi_B, p_B, \dot{x}_B)$.

Sketch of Proof (Case II)

Case II : $\exists(\xi, q, \dot{x}) \in \Omega\ \forall p \leqslant q\ \ (\exists x_0, x_1 \in \mathrm{Eval}(\xi, p, \dot{x})\ x_0 \not{E}\ x_1)$.

We construct a perfect subset of E-inequivalent reals.

For such $(\xi, q, \dot{x}) \in \Omega$, in V,

$$(q, q) \Vdash_{\mathrm{Coll}(\omega,\xi)\times\mathrm{Coll}(\omega,\xi)} \Vdash_{\mathrm{Coll}(\omega,<\kappa)} \dot{x}_{\mathrm{left}} \not{E} \dot{x}_{\mathrm{right}}.$$

Since $\mathcal{P}^V(\mathrm{Coll}(\omega, \xi))$ is countable in $V(\mathbb{R}^{V[G]})$, we can construct a sequence $\langle h_b \mid b \in 2^\omega \rangle$ such that $q \in h_b$ and $h_b \times h_c$ is $\mathrm{Coll}(\omega, \xi) \times \mathrm{Coll}(\omega, \xi)$-generic over V. Then $\{\dot{x}_{h_b} \mid b \in 2^\omega\}$ is a perfect set as desired. $\square$

The dichotomy theorem for sets

As a corollary, we obtain the dichotomy theorem for all sets.

Theorem

In the Solovay model $V(\mathbb{R}^{V[G]})$ for an inaccessible cardinal, for any set X, either X is well-orderable or there is an injection from $\mathbb{R}$ to X.

Sketch of Proof.

We work in $V(\mathbb{R}^{V[G]})$ and let X be a set.

Take an ordinal λ and a surjection $f\colon V_\lambda \times \mathbb{R} \to X$.

For $v \in V_\lambda$ let $X_v := f[\{v\} \times \mathbb{R}]$. Note that $X = \bigcup_{v \in V_\lambda} X_v$.

Assume that there is no injection from $\mathbb{R}$ to X_v for all $v \in V_\lambda$.

For each $v \in V_\lambda$ define an equivalent relation E_v on $\mathbb{R}$ by

$$x\ E_v\ y :\Leftrightarrow f(v, x) = f(v, y).$$

Note that $[x]_{E_v} \mapsto f(v, x)$ is a bijection from $\mathbb{R}/E_v$ to X_v.

Sketch of Proof.

Applying the perfect set dichotomy, we obtain well-order $\leqslant_v$ on $\mathbb{R}/E_v$.

Moreover, by the proof, we can obtain such well-orderings uniformly for $v \in V_\lambda$.

From $\langle \leqslant_v \mid v \in V_\lambda \rangle$ and f, we can construct a well-ordering on $X = \bigcup_{v \in V_\lambda} X_v$. □

Three-element basis theorem for uncountable linear orderings

Theorem (Sakai, T.)

In the Solovay model $V(\mathbb{R}^{V[G]})$ for a weakly compact cardinal, for any uncountable linear ordered set X, at least one of $(\omega_1, <), (\omega_1, >)$ or $(\mathbb{R}, <_{\mathbb{R}})$ is embeddable into X.

Three-element basis theorem for uncountable linear orderings

Theorem (Sakai, T.)

In the Solovay model $V(\mathbb{R}^{V[G]})$ *for a weakly compact cardinal, for any uncountable linear ordered set* X*, at least one of* $(\omega_1, <), (\omega_1, >)$ *or* $(\mathbb{R}, <_{\mathbb{R}})$ *is embeddable into* X*.*

For the proof, we see the partition properties of ω_1 and $\mathbb{R}$ in the Solovay model.

Lemma

In the Solovay model $V(\mathbb{R}^{V[G]})$ for a weakly compact cardinal, $\omega_1 \to (\omega_1)^2_2$ holds.

Lemma (Galvin)

Let $[\mathbb{R}]^2 = P_0 \sqcup P_1$ be a partition. Suppose P_0 and P_1 have the Baire property. Then there is a perfect set $C \subseteq \mathbb{R}$ such that $[C]^2 \subseteq P_i$ for some $i = 0, 1$.

Three-element basis theorem for uncountable linear orderings

Proof. Let $(X, \leqslant_X)$ be an uncountable linear order. Apply the dichotomy theorem.

Case I : X is well-orderable.

Fix an injection $f\colon \omega_1 \to X$. Define $g\colon [\omega_1]^2 \to 2$ by

$$g(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } f(\alpha) <_X f(\beta) \\ 1 & \text{if } f(\alpha) >_X f(\beta) \end{cases}$$

where $\alpha < \beta$.

There is a g-homogeneous $C \subseteq \omega_1$ with $\mathrm{otp}(C) = \omega_1$.

If $g(\{\alpha, \beta\}) = 0$ for all $\{\alpha, \beta\} \in [C]^2$ with $\alpha < \beta$, then $f \restriction C$ is an embedding from $(C, \leqslant)$ to $(X, \leqslant_X)$.

Otherwise, $f \restriction C$ is an embedding from $(C, \geqslant)$ to $(X, \leqslant_X)$.

Three-element basis theorem for uncountable linear orderings

Case II : there is an injection from $\mathbb{R}$ to X.

Fix an injection $f\colon \mathbb{R} \to X$.
By the same argument as Case I and Galvin's lemma, there is a perfect $C \subseteq \mathbb{R}$ such that either $(C, \leqslant_{\mathbb{R}})$ or $(C, \geqslant_{\mathbb{R}})$ is embeddable into $(X, \leqslant_X)$.
Since $(\mathbb{R}, \leqslant_{\mathbb{R}})$ is embeddable into both $(C, \leqslant_{\mathbb{R}})$ and $(C, \geqslant_{\mathbb{R}})$, there is an embedding from $(\mathbb{R}, \leqslant_{\mathbb{R}})$ to $(X, \leqslant_X)$. □

Three-element basis theorem in $L(\mathbb{R})$ satisfying AD

Using Woodin's perfect set dichotomy and partition properties under AD, we can obtain the same consequence in $L(\mathbb{R})$ satisfying AD. [1]

Theorem

Assume ZF $+ V = L(\mathbb{R}) +$ AD. *For any uncountable linear ordered set X, at least one of $(\omega_1, <), (\omega_1, >)$ or $(\mathbb{R}, <_{\mathbb{R}})$ is embeddable into X.*

[1] Weinert independently noticed this result.

Generalized Solovay model

We consider a generalization of the Solovay model for an uncountable regular cardinal μ, collapsing an inaccessible cardinal to the successor of μ.

We investigate subsets of μ^μ in this generalized Solovay model.

Definition

Suppose μ is an uncountable regular cardinal and κ is an inaccessible cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V. Then we call $V((\mu^\mu)^{V[G]}) = \mathsf{HOD}^{V[G]}_{V \cup (\mu^\mu)^{V[G]}}$ the μ-**Solovay model** for κ.

Perfect set property at μ^μ

Definition

Let μ be an uncountable regular cardinal. Suppose T is a subtree of $\mu^{<\mu}$.

- T is *perfect* if T is closed such that for all $s \in T$ there is a splitting node $t \in T$ with $s \subseteq t$.
- $A \subseteq \mu^\mu$ is *perfect* if A is the body $[T]$ of some perfect tree $T \subseteq \mu^{<\mu}$.
- $A \subseteq \mu^\mu$ has the *perfect set property* if $|A| \leqslant \mu$ or A has a perfect subset.

Perfect set property in $V((\mu^\mu)^{V[G]})$

As in the Solovay model $V(\mathbb{R}^{V[G]})$, the perfect set property holds in the generalized Solovay model.

Theorem (Schlicht)

Suppose μ is an uncountable regular cardinal and κ is an inaccessible cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V.

In $V[G]$, every subset of μ^μ in $V((\mu^\mu)^{V[G]})$ has the perfect set property.

Perfect dichotomy in $V((\mu^\mu)^{V[G]})$

We consider the perfect set dichotomy for equivalence relation on μ^μ in $V((\mu^\mu)^{V[G]})$.

Theorem (Sakai, T.)

Suppose μ is an uncountable regular cardinal and κ is an inaccessible cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V.

In the μ-Solovay model $V((\mu^\mu)^{V[G]})$ for κ, for any equivalence relation E on μ^μ, either μ^μ/E is well-orderable or there is a pairwise E-inequivalent perfect subset of μ^μ.

We see the issues that arise in proving this higher analogue.

Quotient of Lévy collpase

One of the most important facts in the analysis of the Solovay model $V(\mathbb{R}^{V[G]})$ is the following factorization.

Fact (Factorization of $\mathrm{Coll}(\omega, {<}\kappa)$)

Suppose κ is an inaccessible cardinal and G is a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter over V. Let $x \in \mathcal{H}_\kappa^{V[G]}$ be a set of ordinals. Then, in $V[G]$, there is a $\mathrm{Coll}(\omega, {<}\kappa)$-generic filter G' over $V[x]$ such that $V[G] = V[x][G']$.

By this fact and the homogeneity of $\mathrm{Coll}(\omega, {<}\kappa)$, whether a real $x \in \mathbb{R}^{V[G]}$ satisfies a given formula in $V[G]$ can be determined by considering a small extension of V containing x.

Quotient of Lévy collpase

Suppose μ is an uncountable regular cardinal.

The point is that a quotient forcing of a μ-closed forcing is not necessarily μ-closed.
Consequently, the higher analogue of the factorization for $\mathrm{Coll}(\mu, {<}\kappa)$ does not hold.

Quotient of Lévy collpase

Suppose μ is an uncountable regular cardinal.

The point is that a quotient forcing of a μ-closed forcing is not necessarily μ-closed. Consequently, the higher analogue of the factorization for $\mathrm{Coll}(\mu, <\kappa)$ does not hold.

To overcome this problem, we focus on the extensions of V of which $V[G]$ is an forcing extension by $\mathrm{Coll}(\mu, <\kappa)$.

Nice extension

Definition

Suppose μ is an uncountable regular cardinal and κ is an inaccessible cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V.

A set of ordinals $x \in \mathcal{H}_\kappa^{V[G]}$ is *nice* if for any $y \in \mathcal{H}_\kappa^{V[G]}$ there is a poset $\mathbb{P} \in \mathcal{H}_\kappa^{V[x]}$ and a $\mathbb{P}$-generic filter g over $V[x]$ such that

- $\mathbb{P}$ is μ-closed in $V[x]$,
- $y \in V[x][g]$.

Note that for each $\xi < \kappa$, $G_\xi = G \cap \mathrm{Coll}(\mu, {<}\xi)$ is nice.

Nice extension

Nice extensiosns have $\mathrm{Coll}(\mu, <\kappa)$ as a quotient in $V[G]$;

Proposition

Suppose μ is an uncountable regular cardinal and κ is an inaccessible cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, <\kappa)$-generic filter over V.

Suppose $x \in \mathcal{H}_\kappa^{V[G]}$ is nice. Then $V[G]$ is an extension of $V[x]$ by $\mathrm{Coll}(\mu, <\kappa)$.

The higher perfect set dichotomy

By focusing on the nice extensions of V, and applying the argument of Schlicht's proof of the perfect set property in $V((\mu^\mu)^{V[G]})$, we can show the higher perfect set dichotomy in $V((\mu^\mu)^{V[G]})$.

Theorem (Restatement)

Suppose μ is an uncountable regular cardinal and κ is an inaccessible cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V.

In the μ-Solovay model $V((\mu^\mu)^{V[G]})$ for κ, for any equivalence relation E on μ^μ, either μ^μ/E is well-orderable or there is a pairwise E-inequivalent perfect subset of μ^μ.

Application for linear orderings

As in the case of the Solovay model $V(\mathbb{R}^{V[G]})$, we obtain a corollary about the basis of linear orderings, using partition properties of μ^+ and μ^μ.

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Lemma

Suppose μ is an uncountable regular cardinal and κ is a weakly compact cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V.
In $V((\mu^\mu)^{V[G]})$, $\mu^+ \to (\mu^+)^2_2$ holds.

Proposition

Suppose μ is an uncountable regular cardinal and κ is an inaccessible cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V.
In $V((\mu^\mu)^{V[G]})$ let $[\mu^\mu]^2 = P_0 \sqcup P_1$ be a partition. Then there is a perfect $C \subseteq \mu^\mu$ such that $[C]^2 \subseteq P_i$ for some $i = 0, 1$.

Three-element basis

Theorem (Sakai, T.)

Suppose μ is an uncountable regular cardinal and κ is a weakly compact cardinal with $\mu < \kappa$. Let G be a $\mathrm{Coll}(\mu, {<}\kappa)$-generic filter over V.

In $V((\mu^\mu)^{V[G]})$, for any linear ordering X such that there is no injection from X to μ, at least one of $(\mu^+, \leqslant)$, $(\mu^+, \geqslant)$ or $(2^\mu, \leqslant_{\mathrm{lex}})$ is embeddable into X, where $\leqslant_{\mathrm{lex}}$ is the lexicographic ordering.

Question : higher analogues

Perfect set property and perfect set dichotomy hold in μ-Solovay model.

However, some higher analogues of regularity properties fail.

Fact

The higher analogue of the Baire property for all subsets of μ^μ in $V((\mu^\mu)^{V[G]})$ fails.

Question

What types of statements have higher analogues, and which ones fail?

Question : generalization at singular

We have focused on the generalization of the Solovay model to uncountable regular cardinals, using $\mathrm{Coll}(\mu, {<}\kappa)$.

Question : generalization at singular

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The generalizations at singular cardinal have been investigated.

Theorem (Dimonte)

Suppose θ is an inaccessible cardinal and μ is a θ-supercompact cardinal with $\mu < \theta$. Let U be a θ-supercompactness measure and let $\mathbb{P}_U$ be the supercompact Prikry forcing for U. Let G be a $\mathbb{P}_U$-generic filter over V. In $V[G]$,

$$H^G := \bigcup\{\mathcal{P}(\mu) \cap V[G \downarrow \alpha] \mid \alpha < \theta\}.$$

Then in $V[G]$, all subsets of ${}^{\omega}\mu$ in $L(H^G)$ have the perfect set property.

Question : generalization at singular

The proof of the perfect set property in $L(H^G)$ is similar to Solovay's original proof.

Question

In the singular generalized Solovay model $L(H^G)$, does the perfect set dichotomy hold for all equivalence relations on ${}^{\omega}\mu$?

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