c.c.c. Filters.

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Talagrand's theorem

Definition

A filter $\mathscr{F} \subseteq \omega^\omega$ is a family of sets closed under supersets and finite intersections. We will assume that $\bigcap \mathscr{F} = \emptyset$. An ultrafilter is a maximal filter.

Theorem [Talagrand]

A filter $\mathscr{F} \subseteq \mathcal{P}(\omega)$ has the Baire property iff is meager iff is bounded, i.e., if there is an interval partition $\langle I_n \rangle$ of ω such that, for all $F \in \mathscr{F}$ and for almost all $n \in \omega$, $F \cap I_n \neq \emptyset$.

Non-meager filters

Basic fact of life

Ultrafilters are nonmeager.

A set A is <u>$\mathscr{F}$-positive</u> if $A \cap F$ is infinite for every $F \in \mathscr{F}$. Equivalently, $A \notin \mathscr{F}^*$.

The algebra of $\mathscr{F}$-positive sets: $\mathcal{P}(\omega)/\mathscr{F}^*$.

We will think of the algebra $\mathcal{P}(\omega)/\mathscr{F}^*$ as the algebra of positive sets.

Observation

A filter $\mathscr{U}$ is an ultrafilter iff $\mathcal{P}(\omega)/\mathscr{U}^*$ has two elements.

c.c.c. Filters

Definition

A filter $\mathscr{F} \subseteq \omega^\omega$ is c.c.c. if $\mathcal{P}(\omega)/\mathscr{F}^*$ is c.c.c., i.e., every antichain of positive sets is countable.

Example: If $\mathscr{U}$ is an ultrafilter, then

$$\mu(A) = \lim_{\mathscr{U}} \frac{|A \cap n|}{n}$$

is a (non-atomic) finite additive measure, that vanishes on points.

Let $\mathcal{N}_\mu = \{A \subseteq \omega : \mu(A) = 0\}$. Then $\mathcal{N}_\mu^*$ is c.c.c.

Observation

c.c.c. filters are non-meager.

Proof: For all infinite A, we have that $\bigcup_{n\in A} I_n \in \mathscr{F}^+$.

Observation

Filters that admit a measure (i.e. filters of the form $\mathcal{N}_\mu^*$) are c.c.c. and every ultrafilter admits a measure.

$$\delta_{\mathscr{U}}(A) = \begin{cases} 1 & \text{if } A \in \mathscr{U}, \\ 0 & \text{otherwise.} \end{cases}$$

Relationships

Observation

Ultrafilter $\to$ Measure $\to$ c.c.c. $\to$ non-meager.

None of these can be reversed.

General Problem

Find properties that help us distinguish these classes of filters.

Cardinal invariants

A basis of a filter $\mathcal{B} \subseteq \mathscr{F}$ is a collection of sets closed under finite intersections such that every $F \in \mathscr{F}$ contains a $B \in \mathcal{B}$.

Theorem [P. Simon]

There is a nonmeager filter $\mathscr{F}$ that has a basis of size $\mathfrak{b}$.

Definition

If φ is a property about filters, then

$$\mathfrak{u}_\varphi = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a basis for a filter with the property } \varphi\}$$

Cardinal Invariants

- $\mathfrak{u}_{\text{nonmeager}}$, where φ is the property of being nonmeager ($= \mathfrak{b}$),
- $\mathfrak{u}_{\text{c.c.c}}$, where φ is the property of being c.c.c.,
- $\mathfrak{u}_{\text{measure}}$, where φ is the property of being the filter that admits a measure,
- $\mathfrak{u}_{\text{ultrafilter}}$, where φ is the property of being an ultrafilter. $(= \mathfrak{u})$.

Bounds

Observation

$\mathfrak{b} = \mathfrak{u}_{\text{nonmeager}} \leq \mathfrak{u}_{\text{c.c.c.}} \leq \mathfrak{u}_{\text{measure}} \leq \mathfrak{u}_{\text{ultrafilter}} = \mathfrak{u}$.

Theorem

$\min\{\mathfrak{d}, \mathfrak{r}\} \leq \mathfrak{u}_{\text{c.c.c.}}$.

If $\mathcal{P}(\omega)/\mathscr{F}$ has an atom A, then $\mathscr{F} \upharpoonright A$ is an ultrafilter over A.

Theorem

$\mathfrak{d} \leq \hat{\mathfrak{u}}_{\text{c.c.c.}}$, where $\hat{\mathfrak{u}}_{\text{c.c.c.}}$ is its non-atomic version.

Consistency results

Theorem

It is consistent that $\text{cov}(\mathcal{N}) > \hat{\mathfrak{u}}_{\text{measure}}$.

Theorem

It is consistent that $\text{cof}(\mathcal{M}) < \hat{\mathfrak{u}}_{\text{c.c.c.}}$.

Theorem

In the Sacks model, $\hat{\mathfrak{u}}_{\text{measure}}$ is small.

Questions

Question

Is $\hat{\mathfrak{u}}_{\text{measure}} \leq \text{cof}(\mathcal{N})$?

What is the value of $\hat{\mathfrak{u}}_{\text{measure}}$ in the Silver model?

Question

Is $\hat{\mathfrak{u}}_{\text{measure}} = \hat{\mathfrak{u}}_{\text{c.c.c.}}$?

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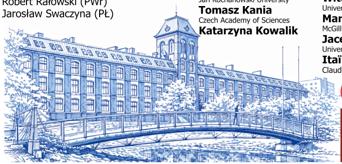

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Thank you for your attention!

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