Ideal zoo in the Baire space II

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Winter School in Abstract Analysis 2026
section Set Theory & Topology
04.02.2026, Hejnice

Definition

Let $h \in \omega^\omega$, $\limsup_n h(n) = \infty$ and $F \subseteq \omega^\omega$. We will say that

- $F \in f\mathcal{N}(h)$ if there is a sequence $(S_n)_{n\in\omega}$, $S_n \subseteq \omega^n$, $\sum_{n\in\omega} \frac{|S_n|}{h(n)} < \infty$, such that
$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x \restriction n \in S_n)\};$$
- $F \in f\mathcal{S}(h)$ if there is a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $\sum \frac{|J_n|}{h(|I_n|)} < \infty$ such that
$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x \restriction I_n \in J_n)\};$$
- $F \in f\mathcal{E}(h)$ if there is a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $\sum \frac{|J_n|}{h(|I_n|)} < \infty$ such that
$$F \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x \restriction I_n \in J_n)\};$$

Replacing h with Fin replaces the series convergence condition with finiteness of S_n and J_n.

$f\mathcal{E}$ vs $f\mathcal{N}$

In 2^ω: $\mathcal{E} \subseteq \mathcal{N}$.

Proposition

Let $h \in \omega^\omega$ satisfy $h(a+b) \geq h(a)h(b)$ for any $a, b \in \omega$. Then $f\mathcal{E}(h) \subseteq f\mathcal{N}(h)$.

Theorem

$f\mathcal{E}(n \mapsto n^2) \not\subseteq f\mathcal{N}(p)$ for any polynomial p.

$f\mathcal{N}$ vs $f\mathcal{S}$

In 2^ω: $\mathcal{S} \subseteq \mathcal{N}$.

Theorem

- *For every* $h \in \omega^\omega$, $\limsup_{n\to\infty} h(n) = \infty$, *there exists* $h' \in \omega^\omega$ *such that* $f\mathcal{N}(h) \subseteq f\mathcal{S}(h')$.
- *For every* $h \in \omega^\omega$. $\limsup_{n\to\infty} h(n) = \infty$, *there exists* $h' \in \omega^\omega$, $\limsup_n h'(n) = \infty$ *such that* $f\mathcal{N}(h') \subseteq f\mathcal{S}(h)$.

Remark

- $f\mathcal{N}(2^n) \not\subseteq f\mathcal{S}(2^n)$.
- *There is* $h \in \omega^\omega$ *for which* $f\mathcal{N}(h) \subseteq f\mathcal{S}(h)$. *(It's* log*)*

Proposition

$f\mathcal{S}(h) \not\subseteq f\mathcal{N}(\mathrm{Fin})$ *for every* $h \in \omega^\omega$, $\limsup_n h(n) = \infty$.

In 2^ω every null set is a union of two small sets (Bartoszyński 1988).

Problem

Let $A \in f\mathcal{N}(h)$. Are there sets $S_0, S_1 \in f\mathcal{S}(h)$ such that $A \subseteq S_0 \cup S_1$? Or at least $f\mathcal{N}(h) \subseteq \sigma(f\mathcal{S}(h))$?

Lemma

Let $s, t \in \omega^\omega$. Assume that

$$(\forall k \in \omega)(\exists N \in \omega)(\sum_{i=k}^{N} s(i) > \sum_{i=1}^{N} t(i)).$$

Then there exists

$$F = \{x \in \omega^\omega : (\exists^\infty n)(x \restriction n \in S_n)\}, \quad |S_n| = s(n),$$

such that for any

$$F' = \{x \in \omega^\omega : (\exists^\infty n)(x \restriction n \in T_n)\}, \quad (\forall^\infty n)(|T_n| \leq t(n)),$$

it is the case that $F \not\subseteq F'$.

Corollary

For every $s \in \omega^\omega$ *there exist* $(S_n)_{n\in\omega}, S_n \subseteq \omega^n, |S_n| = s_n$*, and the set*

$$F = \{x \in \omega^\omega : (\exists^\infty n)(x \restriction n \in S_n)\}$$

such that for any set

$$F' = \{x \in \omega^\omega : (\exists^\infty n)(x \restriction n \in T_n)\}, \quad (\forall^\infty n)(|T_n| < |S_n|),$$

it holds that $F \not\subseteq F'$.

Corollary

Suppose that $\sum_{n\in\omega} \frac{f(n)}{g(n)} < \infty$ *for some* $f, g \in \omega^\omega$. *Then there is* $F \in f\mathcal{N}(g) \setminus f\mathcal{N}(f)$.

Theorem

Let $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. *Then* $f\mathcal{N}(f) \subsetneq f\mathcal{N}(g)$.

Corollary

There is a set $\{f_\alpha \in \omega^\omega : \alpha < \mathfrak{c}\}$ *such that* $f\mathcal{N}(f_\alpha) \subsetneq f\mathcal{N}(f_\beta)$ *or* $f\mathcal{N}(f_\beta) \subsetneq f\mathcal{N}(f_\alpha)$ *for* $\alpha \neq \beta$.

Theorem

There are $f_\alpha \in \omega^\omega, \alpha < \mathfrak{c}$, *such that* $f\mathcal{N}(f_\alpha) \not\subseteq f\mathcal{N}(f_\beta)$ *for* $\alpha \neq \beta$.

Łukasz's talk: $f\mathcal{N}(h) \perp \mathcal{M}$. By previous results also $f\mathcal{S}(h) \perp \mathcal{M}$. What about $\mathcal{M}_-$?

Definition

$F \in \mathcal{M}_-$ if there are $x_F \in \omega^\omega$ and a partition of ω into intervals $(I_n)_{n\in\omega}$ such that

$$F \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x \restriction I_n \neq x_F \restriction I_n)\}.$$

Theorem

- $f\mathcal{S}(h) \perp \mathcal{M}_-$ *for every* $h \in \omega^\omega$, $\limsup_n h(n) = \infty$.
- $f\mathcal{N}(\mathrm{Fin}) \not\perp \mathcal{M}_-$.

Thank you for your attention!

Mazurkiewicz Łukasz, Michalski Marcin, Zeberski *Szymon, An Ideal Zoo in the Baire Space*, arXiv: 2510.27435.