# *Everywhere J sets*

Jakub Cieplechowicz

Wrocław University of Science and Technology

$$\mathcal{EM} = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega) A \restriction S \text{ is meager } 2^S\},$$

$$\mathcal{EN} = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega) A \restriction S \text{ is null } 2^S\},$$

$$\mathbb{B}_2 = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega) A \restriction S \neq 2^S\},$$

$$\mathcal{I}_0 = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega)(\exists S' \in [S]^\omega)|A \restriction S'| \leq \omega\}.$$

**CH** $\checkmark$

**ZFC?**

$$\begin{array}{ccccc} & & \mathcal{EM} & & \\ & \not\subseteq & & \not\supseteq & \\ \mathcal{I}_0 & & \not\subseteq \; \not\supseteq & & \mathbb{B}_2 \\ & \not\supseteq & & \not\subseteq & \\ & & \mathcal{EN} & & \end{array}$$

# ***Perfect*** *here but not here...*

$P \in Perfect(2^\omega) \iff P = [T]$, $T$ perfect tree

$T \subseteq 2^{<\omega}, \sigma \in T$:

- $\mathrm{succ}_T(\sigma) = \{a \in 2 : \sigma^\frown a \in T\}$
- $\mathrm{split}(T) = \{\sigma \in T : |\mathrm{succ}_T(\sigma)| = 2\}$

- perfect: $(\forall \sigma \in T)(\exists \tau \supseteq \sigma)(\tau \in \mathrm{split}(T))$
- uniformly perfect:

perfect & $(\forall n \in \omega)(2^n \cap T \subseteq \mathrm{split}(T) \;\vee\; 2^n \cap \mathrm{split}(T) = \emptyset)$

$T \in \mathcal{UP} \iff T$ is uniformly perfect

$A \in \mathbb{B}_2 \iff (\forall S \in [\omega]^\omega) A \restriction S \neq 2^S$

***Theorem 1.*** $T \in \mathcal{UP} \implies [T] \notin \mathbb{B}_2$.

*Proof.*

$$S = \{n \in \omega : 2^n \cap T \subseteq \mathrm{split}(T)\}.$$

- $|S| = \omega \quad \& \quad [T] \restriction S = 2^S$.

$\blacksquare$

$T$ has *unique splits* ($\in \mathcal{US}$) if

$$\forall n \quad |2^n \cap \mathrm{split}(T)| \leq 1.$$

$$A \in \mathcal{I}_0 \iff (\forall S \in [\omega]^\omega)(\exists S' \in [S]^\omega)|A \restriction S'| \leq \omega$$

***Theorem 2.*** $(\exists T \in \mathcal{US}) \quad [T] \in \mathcal{I}_0.$

$T$ has *unique splits* ($\in \mathcal{US}$) if

$$\forall n \quad |2^n \cap \mathrm{split}(T)| \leq 1.$$

$A \in \mathcal{I}_0 \iff (\forall S \in [\omega]^\omega)(\exists S' \in [S]^\omega)|A \restriction S'| \leq \omega$

***Theorem 2.*** $(\exists T \in \mathcal{US}) \quad [T] \in \mathcal{I}_0$.

*Proof.* $n_\sigma = (1^\frown \sigma)_2 = \sum_{i \in \mathrm{dom}(\sigma)} i \cdot \sigma(i) + 2^{|\sigma|}$, e.g.
$n_{10} = (1^\frown 10)_2 = 2^0 \cdot 0 + 2^1 \cdot 1 + 2^2 \cdot 1 = 6.$

$\{t_\sigma \in 2^{<\omega} : \sigma \in 2^{<\omega}\}$:

(1) $t_\emptyset = \emptyset$,

(2) $\forall \sigma \quad t_{\sigma^\frown i} = t_\sigma{}^\frown \underbrace{0^\frown \ldots {}^\frown 0}_{n_\sigma}{}^\frown i$

$(\star)\ |t_\sigma| = |t_\eta| \iff (\exists \tau)(\tau^\frown 1 = \sigma \ \wedge \ \tau^\frown 0 = \eta)$

"$\mathit{split} = 1$"

$S \in [\omega]^\omega$, $\mathrm{p\_splits}^S(\sigma) = \{n \in S : (\exists t \in T)\ t \supseteq \sigma \wedge t \text{ } \mathit{splits\ at}\ n\}$.

- $\bullet$ $a_n \in S$
- $\bullet\bullet$ $\sigma_n \in T$
- $\bullet\bullet\bullet$ $S_n \subseteq S$

(0) $a_0 = \min S$, $V_0 = T \cap 2^{a_0+1}$.

$\sigma, \sigma' \in V_0, \sigma \neq \sigma' \implies \mathrm{p\_splits}^S(\sigma) \cap \mathrm{p\_splits}^S(\sigma') = \emptyset\ (\star)$.

**Case 0.** $\forall \sigma \in V_0 \quad |\text{p\_splits}^S(\sigma)| < \omega \ \rightsquigarrow \ [T] \upharpoonright S$ is finite.

**Case 1.** $\exists \sigma_0 \in V_0 \quad |\text{p\_splits}^S(\sigma_0)| = \omega$.

$S_0 = \text{p\_splits}^S(\sigma_0)$, $a_1 = \min S_0$,
$V_1 = \{t \in T : t \supseteq \sigma_0 \ \wedge \ t \in 2^{a_1+1}\}$.

**Case 0.** $\forall \sigma \in V_0 \quad |\text{p\_splits}^S(\sigma)| < \omega \ \rightsquigarrow \ [T] \upharpoonright S$ is finite.

**Case 1.** $\exists \sigma_0 \in V_0 \quad |\text{p\_splits}^S(\sigma_0)| = \omega.$

$S_0 = \text{p\_splits}^S(\sigma_0)$, $a_1 = \min S_0$,
$V_1 = \{t \in T : t \supseteq \sigma_0 \ \wedge \ t \in 2^{a_1+1}\}$.

$\exists \sigma_1 \in V_1 \quad |\text{p\_splits}^S(\sigma_1)| = \omega.$

$S_1 = \text{p\_splits}^S(\sigma_1)$, $a_2 = \min S_1$,
$V_2 = \{t \in T : t \supseteq \sigma_1 \ \wedge \ t \in 2^{a_2+1}\}$

$S_n = \text{p\_splits}^S(\sigma_n)$, $a_{n+1} = \min S_n$,
$V_{n+1} = \{t \in T : t \supseteq \sigma_n \ \wedge \ t \in 2^{a_{n+1}+1}\}$.

$A = \{a_n : n \in \omega\} \subseteq S, \ |A| = \omega$

$$[T] \restriction A = \bigcup_n \bigcup_{\sigma \in V_n \setminus \{\sigma_n\}} \big([T] \cap [\sigma]\big) \restriction A \cup \{\bigcup_n \sigma_n \restriction A\}.$$

$S_n = \text{p\_splits}^S(\sigma_n)$, $a_{n+1} = \min S_n$,
$V_{n+1} = \{t \in T : t \supseteq \sigma_n \ \wedge \ t \in 2^{a_{n+1}+1}\}$.

$A = \{a_n : n \in \omega\} \subseteq S, \ |A| = \omega$

$$[T] \restriction A = \bigcup_n \bigcup_{\sigma \in V_n \setminus \{\sigma_n\}} \big([T] \cap [\sigma]\big) \restriction A \cup \{\bigcup_n \sigma_n \restriction A\}.$$

$([T] \cap [\sigma]) \restriction A$ is finite for $\sigma \in V_n \setminus \{\sigma_n\}$,

$t \in ([T] \cap [\sigma]) \restriction A \implies (\forall k \in A)(k > a_n \to t(k) = 0)$,

$[T] \restriction A \in \mathcal{I}_0$.

$\blacksquare$

***Theorem 2.*** $(\exists H \in \mathcal{US}) \quad [H] \notin \mathbb{B}_2$.

$$\forall \sigma \quad t_{\sigma^\frown i} = t_\sigma{}^\frown \underbrace{i^\frown \ldots {}^\frown i}_{n_\sigma + i}$$

$T \subseteq 2^{<\omega}$ is a splitting tree ($\in \mathcal{ST}$) if

$$(\forall t \in T)(\exists N \in \omega)(\forall n \geq N)(\forall i \in 2)(\exists t' \in T \cap 2^{n+1})(t \subseteq t' \wedge t'(n) = i).$$

***Theorem 3.*** $(\forall T \in \mathcal{ST}) \quad [T] \notin \mathbb{B}_2$

***Theorem 3.*** $(\forall T \in \mathcal{ST}) \quad [T] \notin \mathbb{B}_2$

*Proof.* $N_\sigma$ minimal splitting point for $\sigma \in T$:

$$(\forall n \geq N_\sigma)(\forall i \in 2)(\exists \sigma' \in T \cap 2^{n+1})(\sigma \subseteq \sigma' \wedge \sigma'(n) = i).$$

$\{s_i : i \in \omega\} \in [\omega]^\omega$

- $s_0 = 0$, $\sigma_\emptyset(s_0) = k$
- $s_1 = N_{\sigma_\emptyset}$
- $\sigma_0, \sigma_1 \supseteq \sigma_\emptyset$, $\sigma_0(N_{\sigma_\emptyset}) = 0$, $\sigma_1(N_{\sigma_\emptyset}) = 1$,
  $s_2 = \max\{N_{\sigma_1}, N_{\sigma_0}\}$.
- $s_3 = \max\{N_{\sigma_{11}}, N_{\sigma_{10}}, N_{\sigma_{01}}, N_{\sigma_{00}}\}$.

$\vdots$

- $s_{n+1} = \max\{N_{\sigma_j} : |j| = n\}$.

$S = \{s_i : i \in \omega\} \rightsquigarrow [T] \restriction S = 2^S$.

∎

$$\begin{array}{ccccc} & & \mathcal{EM} & & \\ & \not\subseteq & & \not\supseteq & \\ \mathcal{I}_0 & & \not\subseteq \not\supseteq & & \mathbb{B}_2 \\ & \not\supseteq & & \not\subseteq & \\ & & \mathcal{EN} & & \end{array}$$

***Corollary 4.*** We can not distinguish the $\sigma$-ideals $\mathcal{EM}, \mathcal{EN}, \mathcal{I}_0, \mathbb{B}_2$ based on **splitting trees**.

## $\mathcal{EM}, \mathbb{B}_2$

$P \in Perfect(2^\omega) \implies P \restriction S$ closed, $S \in [\omega]^\omega$.

***Corollary 5.*** $\mathcal{EM} \cap Perfect(2^\omega) = \mathbb{B}_2 \cap Perfect(2^\omega)$.

*Proof.*
$P \in Perfect(2^\omega) \setminus \mathcal{EM} \implies (\exists S \in [\omega]^\omega) \quad \mathrm{int}_{2^S}(P \restriction S) \neq \emptyset.$

$P \restriction S \supseteq [\sigma], \sigma \in 2^{<S} \implies P \restriction (S \setminus \mathrm{dom}(\sigma)) = 2^{S \setminus \mathrm{dom}(\sigma)}$, so
$P \notin \mathbb{B}_2.$

$\blacksquare$

*Thank you*