

Everywhere J sets

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$$\mathcal{EM} = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega) A \restriction S \text{ is meager } 2^S\},$$

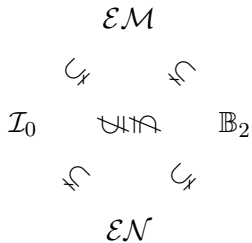
$$\mathcal{EN} = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega) A \restriction S \text{ is null } 2^S\},$$

$$\mathbb{B}_2 = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega) A \restriction S \neq 2^S\},$$

$$\mathcal{I}_0 = \{A \subseteq 2^\omega : (\forall S \in [\omega]^\omega) (\exists S' \in [S]^\omega) |A \restriction S'| \leq \omega\}.$$

CH ✓

ZFC?



Perfect here but not here...

$P \in \text{Perfect}(2^\omega) \iff P = [T], T \text{ perfect tree}$

$T \subseteq 2^{<\omega}, \sigma \in T$:

- $\text{succ}_T(\sigma) = \{a \in 2 : \sigma \frown a \in T\}$
- $\text{split}(T) = \{\sigma \in T : |\text{succ}_T(\sigma)| = 2\}$

- perfect: $(\forall \sigma \in T)(\exists \tau \supseteq \sigma)(\tau \in \text{split}(T))$
- uniformly perfect:

perfect & $(\forall n \in \omega)(2^n \cap T \subseteq \text{split}(T) \vee 2^n \cap \text{split}(T) = \emptyset)$

$T \in \mathcal{UP} \iff T$ is uniformly perfect

$A \in \mathbb{B}_2 \iff (\forall S \in [\omega]^\omega) A \upharpoonright S \neq 2^S$

Theorem 1. $T \in \mathcal{UP} \implies [T] \notin \mathbb{B}_2$.

Proof.

$$S = \{n \in \omega : 2^n \cap T \subseteq \text{split}(T)\}.$$

- $|S| = \omega$ & $[T] \upharpoonright S = 2^S$.



T has *unique splits* ($\in \mathcal{US}$) if

$$\forall n \quad |2^n \cap \text{split}(T)| \leq 1.$$

$$A \in \mathcal{I}_0 \iff (\forall S \in [\omega]^\omega)(\exists S' \in [S]^\omega) |A \upharpoonright S'| \leq \omega$$

Theorem 2. $(\exists T \in \mathcal{US}) \quad [T] \in \mathcal{I}_0.$

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Theorem 2. $(\exists T \in \mathcal{US}) \quad [T] \in \mathcal{I}_0$.

Proof. $n_\sigma = (1 \frown \sigma)_2 = \sum_{i \in \text{dom}(\sigma)} i \cdot \sigma(i) + 2^{|\sigma|}$, e.g.
 $n_{10} = (1 \frown 10)_2 = 2^0 \cdot 0 + 2^1 \cdot 1 + 2^2 \cdot 1 = 6$.

$$\{t_\sigma \in 2^{<\omega} : \sigma \in 2^{<\omega}\}:$$

$$(1) \quad t_\emptyset = \emptyset,$$

$$(2) \quad \forall \sigma \quad t_{\sigma \frown i} = t_\sigma \frown \underbrace{0 \frown \dots \frown 0}_{n_\sigma} \frown i$$

$$(\star) \quad |t_\sigma| = |t_\eta| \iff (\exists \tau)(\tau \smallfrown 1 = \sigma \wedge \tau \smallfrown 0 = \eta)$$

$$“split = 1”$$

$$S \in [\omega]^\omega, \text{ p_splits}^S(\sigma) = \{n \in S : (\exists t \in T) \ t \supseteq \sigma \wedge t \text{ splits at } n\}.$$

- $a_n \in S$
- $\sigma_n \in T$
- $S_n \subseteq S$

$$(0) \ a_0 = \min S, \ V_0 = T \cap 2^{a_0+1}.$$

$$\sigma, \sigma' \in V_0, \sigma \neq \sigma' \implies \text{p_splits}^S(\sigma) \cap \text{p_splits}^S(\sigma') = \emptyset \ (\star).$$

Case 0. $\forall \sigma \in V_0 \quad |\text{p_splits}^S(\sigma)| < \omega \rightsquigarrow [T] \upharpoonright S \text{ is finite.}$

Case 1. $\exists \sigma_0 \in V_0 \quad |\text{p_splits}^S(\sigma_0)| = \omega.$

$S_0 = \text{p_splits}^S(\sigma_0), a_1 = \min S_0,$
 $V_1 = \{t \in T : t \supseteq \sigma_0 \wedge t \in 2^{a_1+1}\}.$

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$$\exists \sigma_1 \in V_1 \quad |\text{p_splits}^S(\sigma_1)| = \omega.$$

$$S_1 = \text{p_splits}^S(\sigma_1), \quad a_2 = \min S_1, \\ V_2 = \{t \in T : t \supseteq \sigma_1 \wedge t \in 2^{a_2+1}\}$$

$$S_n = \text{p_splits}^S(\sigma_n), \ a_{n+1} = \min S_n, \\ V_{n+1} = \{t \in T : t \supseteq \sigma_n \ \wedge \ t \in 2^{a_{n+1}+1}\}.$$

$$A = \{a_n : n \in \omega\} \subseteq S, \ |A| = \omega$$

$$[T] \restriction A = \bigcup_n \bigcup_{\sigma \in V_n \setminus \{\sigma_n\}} ([T] \cap [\sigma]) \restriction A \cup \{\bigcup_n \sigma_n \restriction A\}.$$

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$$([T] \cap [\sigma]) \restriction A \text{ is finite for } \sigma \in V_n \setminus \{\sigma_n\},$$

$$t \in ([T] \cap [\sigma]) \restriction A \implies (\forall k \in A)(k > a_n \rightarrow t(k) = 0),$$

$$[T] \restriction A \in \mathcal{I}_0.$$



Theorem 2. $(\exists H \in \mathcal{US}) \quad [H] \notin \mathbb{B}_2.$

$$\forall \sigma \quad t_{\sigma \smallfrown i} = t_{\sigma} \smallfrown \underbrace{i \smallfrown \dots \smallfrown i}_{n_{\sigma} + i}$$

$T \subseteq 2^{<\omega}$ is a splitting tree ($\in \mathcal{ST}$) if

$$(\forall t \in T)(\exists N \in \omega)(\forall n \geq N)(\forall i \in 2)(\exists t' \in T \cap 2^{n+1})(t \subseteq t' \wedge t'(n) = i).$$

Theorem 3. $(\forall T \in \mathcal{ST}) \quad [T] \notin \mathbb{B}_2$

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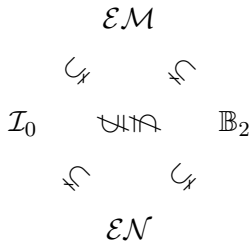
Proof. N_σ minimal splitting point for $\sigma \in T$:

$$(\forall n \geq N_\sigma)(\forall i \in 2)(\exists \sigma' \in T \cap 2^{n+1})(\sigma \subseteq \sigma' \wedge \sigma'(n) = i).$$

$$\{s_i : i \in \omega\} \in [\omega]^\omega$$

- $s_0 = 0, \sigma_\emptyset(s_0) = k$
- $s_1 = N_{\sigma_\emptyset}$
- $\sigma_0, \sigma_1 \supseteq \sigma_\emptyset, \sigma_0(N_{\sigma_\emptyset}) = 0, \sigma_1(N_{\sigma_\emptyset}) = 1,$
 $s_2 = \max\{N_{\sigma_1}, N_{\sigma_0}\}.$
- $s_3 = \max\{N_{\sigma_{11}}, N_{\sigma_{10}}, N_{\sigma_{01}}, N_{\sigma_{00}}\}.$
- $\vdots$
- $s_{n+1} = \max\{N_{\sigma_j} : |j| = n\}.$

$$S = \{s_i : i \in \omega\} \rightsquigarrow [T] \restriction S = 2^S.$$



Corollary 4. We can not distinguish the σ -ideals $\mathcal{EM}, \mathcal{EN}, \mathcal{I}_0, \mathbb{B}_2$ based on **splitting trees**.

$\mathcal{EM}, \mathbb{B}_2$

$P \in \text{Perfect}(2^\omega) \implies P \restriction S \text{ closed, } S \in [\omega]^\omega.$

Corollary 5. $\mathcal{EM} \cap \text{Perfect}(2^\omega) = \mathbb{B}_2 \cap \text{Perfect}(2^\omega).$

Proof.

$P \in \text{Perfect}(2^\omega) \setminus \mathcal{EM} \implies (\exists S \in [\omega]^\omega) \quad \text{int}_{2^S}(P \restriction S) \neq \emptyset.$

$P \restriction S \supseteq [\sigma], \sigma \in 2^{<S} \implies P \restriction (S \setminus \text{dom}(\sigma)) = 2^{S \setminus \text{dom}(\sigma)}, \text{ so}$
 $P \notin \mathbb{B}_2.$



Thank you