# Slim sets in vector spaces

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Winter School in Abstract Analysis
31 January – 7 February, 2025, Hejnice

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**Definition**

Let $\mathbb{K}$ be a field and let $V$ be a vector space over $\mathbb{K}$. A set $S \subseteq V$ is called $n$-slim ($n > 0$) if there exists a linear operator $f \colon V \to \mathbb{K}^n$ that preserves affine independence among at most $(n+1)$-element subsets of $S$.

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**Definition**

If $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$ and $V$ is a normed space, we have the obvious notion of continuous $n$-slimness.

**Theorem 1**

Assume $\mathbb{K}$ is an infinite field and $V$ is a vector space over $\mathbb{K}$. Then every set $S \subseteq V$ with $|S| < |\mathbb{K}|$ is 1-slim.

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**Lemma**

*Let $V$ be a vector space over $\mathbb{K}$ and let $S \subseteq V \setminus \{0\}$ be such that $|S| < |\mathbb{K}|$. Then there exists $f \in V^*$ such that $S \cap \ker f = \emptyset$.*

## Lemma

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## Proof.

By Zorn's lemma, there exists a maximal linear subspace $W \leqslant V$ satisfying $W \cap S = \emptyset$. It suffices to show that $W$ has codimension one in $V$. Suppose otherwise and fix linearly independent vectors $a, b \in V$ such that $\operatorname{lin}\{a, b\} \cap W = \{0\}$. Given $\lambda \in \mathbb{K}$, define

$$W_\lambda = W \oplus \mathbb{K}(a + \lambda b).$$

Then $W_\lambda$ is a linear subspace of $V$ properly containing $W$, therefore $(W_\lambda \setminus W) \cap S \neq \emptyset$. We shall prove that $W_\lambda \cap W_\delta = W$ whenever $\lambda \neq \delta$, thus showing that the family $\{(W_\lambda \setminus W)\}_{\lambda \in \mathbb{K}}$ is pairwise disjoint and consequently the set $S$ must have cardinality $\geqslant |\mathbb{K}|$, which is a contradiction.
Suppose $v \in W_\lambda \cap W_\delta \setminus W$. Then $v = w_0 + r_0(a + \lambda b)$ and $v = w_1 + r_1(a + \delta b)$ for some $w_0, w_1 \in W$ and $r_0, r_1 \in \mathbb{K}$. Hence

$$w_0 - w_1 = (r_1 - r_0)a + (r_1\delta - r_0\lambda)b,$$

therefore both sides of this equation are zero. Furthermore, $r_0 = r_1$ and $r_0\lambda = r_1\delta$. It cannot be the case that $r_0 = 0 = r_1$, because $v \notin W$. We conclude that $\lambda = \delta$. □

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**Proposition**

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**Proof.**

Every bounded functional on $H$ has a countable support with respect to $S$. $\square$

**Proposition**

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**Proof.**

Apply the Baire Category Theorem. □

### Theorem 2

In a separable normed space, every set of cardinality $< \mathfrak{c}$ is continuously $k$-slim for every $k > 0$.

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## Proof of the case $k = 1$.

Fix a set $X$ in a real or complex separable normed space $V$. Let $Y = X - X$. Given $y \in Y \setminus \{0\}$, let

$$A_y = \{\varphi \in V' : \varphi(y) = 0\}.$$

It suffices to show that $\bigcup_{y \in Y} A_y \neq V'$. We now follow the argument of Klee [2] (see also [1]). Choose a weak* dense set $\{b_n\}_{n \in \omega}$ in the unit ball of $V'$. Given $\lambda \in (0, 1)$, define

$$b_\lambda = \sum_{n=0}^{\infty} \lambda^n b_n.$$

This is well-defined, because the series on the right-hand side is absolutely convergent. Fix $v \in V$ and define

$$f_v(\lambda) = b_\lambda(v) = \sum_{n=0}^{\infty} \lambda^n b_n(v).$$

If $v \neq 0$, $f_v$ is not constant zero. Furthermore, $f_v$ is an analytic function on the interval $(0, 1 - \varepsilon)$, where $0 < \varepsilon < 1$ is fixed, therefore it may have only finitely many zeros. It follows that each set of the form

$$\ker(v) = \{\varphi \in V' : \varphi(v) = 0\}, \qquad v \neq 0$$

can contain only finitely many vectors of the form $b_\lambda$, where $0 < \lambda < 1 - \varepsilon$. Note that $\ker(y) = A_y$, defined above. This completes the proof. □

**Corollary**

*In a vector space over* $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, *every subset of cardinality* $< \mathfrak{c}$ *is* $k$*-slim for every* $k > 0$.

D. Głodkowski, P. Koszmider, *On coverings of Banach spaces and their subsets by hyperplanes*, Proc. Amer. Math. Soc. 150 (2022) 817–831

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