Flat sets in vector spaces

Wiesław Kubiś

February 5, 2026

Abstract

Let $\mathbb{K}$ be a field and let V be a vector space over $\mathbb{K}$. A set $S \subseteq V$ is called *n-slim* ($n > 0$) if there exists a linear operator $f \colon V \to \mathbb{K}^n$ that preserves affine independence among at most $(n+1)$-element subsets of S. In particular, S is 1-slim if only if there exists a linear functional on V that is one-to-one on S.

We show that if $\mathbb{K}$ is infinite then every set of cardinality $< |\mathbb{K}|$ is 1-slim. The argument is elementary, using basic linear algebra. On the other hand, we do not know whether the same is true for n-slimness when $n > 1$.

On the other hand, we prove that every subset S of a separable normed space over the real or complex numbers is *continuously* n-slim for every $n > 0$, as long as $|S| < \mathfrak{c}$, i.e., there exists a bounded operator witnessing n-slimness. The proof is based on an old trick of V. Klee, using analytic functions. Easy examples show that this fails in non-separable Banach spaces.

(Joint work with Wojciech Bielas)