Topology idealized

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Motivation, Cale's work

Aleksandar Pavlovic (1974-2025)

A. Njamcul, A. Pavlović, On topology expansion using ideals. Topology Appl. 374 (2025)

Definition

(X, τ) a topological space and $\mathcal{I} \subseteq P(X)$ an ideal.

- $A^* = \{x \in X : \ (\forall U \in \tau)(x \in U \to U \cap A \notin \mathcal{I})\}$.
- $A \mapsto A^*$ is called local function.
- $cl^*(A) = A \cup A^*$ is a closure operator.
- $\tau_{\mathcal{I}}$ is a topology on X given by cl^*.

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- $\tau_{\mathcal{I}}$ is a topology on X given by cl^*.

Fact

The basis of topology $\tau_{\mathcal{I}}$ is given by sets of the form $U \setminus I$, where $U \in \tau$ and $I \in \mathcal{I}$.

Density

$d(X, \tau) = \min\{|D| : D \subseteq X \text{ is dense in } X\}$

Fact

$\text{non}(\mathcal{I}) \leq d(X, \tau_{\mathcal{I}})$.

Proof.

Assume D is dense and $|D| < \text{non}(\mathcal{I})$.
Then $D \in \mathcal{I}$, so $X \setminus D \in \tau_{\mathcal{I}}$. $\square$

Weight

$w(X, \tau) = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \tau \text{ is a topological base } \tau\}.$

Tightness

$t(X, \tau) = \min\{\kappa : \forall A \subseteq X\ \forall x \in \mathrm{cl}A\ \exists B \subseteq A\ |B| \leq \kappa \wedge x \in \mathrm{cl}B\}.$

Lindelof number

$L(X, \tau) = \min\{\kappa : \forall \mathcal{U} \subseteq \tau\ (X = \bigcup \mathcal{U} \to \exists \mathcal{V} \in [\mathcal{U}]^{\leq \kappa}\ X = \bigcup \mathcal{V})\}.$

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Theorem

Assume that $|A| \leq t(X,\tau)$ implies $A \in \mathcal{I}$. Then

$$\mathrm{cov}(\mathcal{I}) \leq \mathrm{w}(\mathrm{X}, \tau_{\mathcal{I}}) + \mathrm{L}(\mathrm{X}, \tau).$$

Theorem

$[X]^{\leq t(X)} \subseteq \mathcal{I}$. Then $\mathrm{cov}(\mathcal{I}) \leq \mathrm{w}(\mathrm{X}, \tau_{\mathcal{I}}) + \mathrm{L}(\mathrm{X}, \tau)$.

Proof.

- $\mathcal{B} = \{U_\alpha \backslash A_\alpha : \alpha < w(X, \tau_{\mathcal{I}})\}$.
- $\mathcal{A} = \{A_\alpha : \alpha < w(X, \tau_{\mathcal{I}})\}$.
- $|X \setminus \bigcup \mathcal{A}| < t(X, \tau) + L(X, \tau)$.

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- $|X \setminus \bigcup \mathcal{A}| < t(X, \tau) + L(X, \tau)$.
  - Assume oposite.
  - there is $E \subseteq X \setminus \bigcup \mathcal{A}$, $x \in cl(E) \setminus E$, $E \in \mathcal{I}$
  - $X \setminus E = \bigcup \{U_\alpha \setminus A_\alpha : \alpha \in S\}$.
  - $U = \bigcup \{U_\alpha : \alpha \in S\}$.
  - $U \cap E = \emptyset$;

$$x' \in U_{\alpha_0} \setminus A_{\alpha_0} \subseteq \bigcup \{U_\alpha \setminus A_\alpha : \alpha \in S\} \subseteq X \setminus \{x'\}.$$

  - Then $X \setminus E = U$. Contradiction.

Character

$\chi(X,\tau) = \mathsf{min}\{\kappa : (\forall x \in X)(\exists \mathcal{B} \in [\tau]^{\leq\kappa})(\mathcal{U}$ is a base at $x)\}$.

Theorem

$t(X,\tau_{\mathcal{I}}) \leq \mathrm{cof}(\mathcal{I}) \cdot \chi(\mathrm{X},\tau)$.

Continuous functions

Theorem

Let $\mathcal{I} = [\mathbb{R}]^{\leq\omega}$, τ - standard topology on $\mathbb{R}$.
If $f : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau_{\mathcal{I}})$ is continuous then f is constant.

Proof.

Assume that $a < b$ and $f(a) \neq f(b)$.

$$g(x) = \begin{cases} f(a), & \text{if } x \leq a, \\ f(x), & \text{if } x \in (a, b), \\ f(b), & \text{if } x \geq b. \end{cases}$$

Proof...

- $x_0 \in \mathbb{R}$ such that $f(a) \neq g(x_0) \neq f(b)$.
  $X_0 = g^{-1}[\{g(x_0)\}]$, $m_0 = \max X_0$.

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- $x_{n+1} \in (m_n, b)$ such that $f(a) \neq g(x_{n+1}) \neq f(b)$.
  $X_{n+1} = g^{-1}[\{g(x_{n+1})\}]$, $m_{n+1} = \max X_{n+1}$.

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- $m = \lim_{n\to\infty} m_n$, $m \notin \bigcup_{n\in\omega} X_n$.
  $Y = \{f(x_n) : \ n \in \omega\}$, $f(m) \notin Y$

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  $Y = \{f(x_n) : \ n \in \omega\}$, $f(m) \notin Y$
- Y is countable, hence $\tau_{\mathcal{I}}$-closed, but $f^{-1}[Y]$ is not closed.
  Contradiction.

Theorem

Let(X, τ_X) be a Polish space, $\mathcal{I} = \{A \subseteq X : |A| \leq \omega\}$.
Every continuous function $f : (2^\omega, \tau) \to (X, \tau_{X\mathcal{I}})$ has finitely many values.

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Thank you for your attention!