Ideal zoo in the Baire space 1

Łukasz Mazurkiewicz
Marcin Michalski
Szymon Żeberski

Wrocław University of Science and Technology
lukasz.mazurkiewicz@pwr.edu.pl

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Fake category

Fake meager

Theorem (Bartoszyński, Judah)

Let $F \subseteq 2^\omega$. $F \in \mathcal{M}$ if there are $x_F \in 2^\omega$ and a partition of ω into intervals $(I_n)_{n\in\omega}$ such that

$$F \subseteq \{x \in 2^\omega : (\forall^\infty n)(x\restriction I_n \neq x_F\restriction I_n)\}.$$

Definition

Let $F \subseteq \omega^\omega$. $F \in \mathcal{M}_-$ if there are $x_F \in \omega^\omega$ and a partition of ω into intervals $(I_n)_{n\in\omega}$ such that

$$F \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x\restriction I_n \neq x_F\restriction I_n)\}.$$

Observations

- $\mathcal{M}_-$ is a translation invariant $\sigma-$ideal
- $\mathcal{M}_- \subsetneq \mathcal{M}$

Chain conditions

Definition

$\mathcal{I}$ is $\kappa-$cc if for every family of Borel $\mathcal{I}-$positive sets $\{A_\alpha : \alpha < \kappa\}$ there are $\alpha \neq \beta$ such that $A_\alpha \cap A_\beta \notin \mathcal{I}$.

Theorem

$\mathcal{M}_-$ is not $\mathrm{add}(\mathcal{M})$*−cc.*

Chain conditions

Definition

$\mathcal{I}$ is $\kappa-$cc if for every family of Borel $\mathcal{I}-$positive sets $\{A_\alpha : \alpha < \kappa\}$ there are $\alpha \neq \beta$ such that $A_\alpha \cap A_\beta \notin \mathcal{I}$.

Theorem

$\mathcal{M}_-$ is not add$(\mathcal{M})$*−cc.*

Lemma

For every $A \in \mathcal{M}$ there is a nowhere dense set $B \notin \mathcal{M}_-$ such that $A \cap B = \emptyset$.

Question

Is $\mathcal{M}_-$ $\mathfrak{c}-$cc?

Another meager-like ideals

Definition (Newelski, Rosłanowski)

For $f : \omega^{<\omega} \to \omega$ let

$$D_f = \{x \in \omega^\omega : (\forall^\infty n)(x(n) \neq f(x{\restriction}n))\}.$$

Denote $\mathcal{D}_\omega = \{A \subseteq \omega^\omega : A \subseteq D_f$ for some f$\}$.

Definition (Spinas)

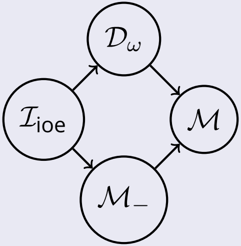
For $y \in \omega^\omega$ let

$$K_y = \{x \in \omega^\omega : (\forall^\infty n)(y(n) \neq x(n))\}.$$

Denote by $\mathcal{I}_{\text{ioe}}$ the $\sigma-$ideal generated by K_y.

Cardinal invariants

Theorem



Theorem (Khomskii, Laguzzi)

$\mathrm{cov}(\mathcal{I}_{\mathrm{ioe}}) = \mathrm{cov}(\mathcal{D}_\omega) = \mathrm{cov}(\mathcal{M})$ *and* $\mathrm{non}(\mathcal{I}_{\mathrm{ioe}}) = \mathrm{non}(\mathcal{D}_\omega) = \mathrm{non}(\mathcal{M})$.

Corollary

$\mathrm{cov}(\mathcal{M}_-) = \mathrm{cov}(\mathcal{M})$ *and* $\mathrm{non}(\mathcal{M}_-) = \mathrm{non}(\mathcal{M})$.

Fake measure

Measure in the Cantor space

Definition

Let $F \subseteq 2^\omega$. F is null if

$$(\forall \varepsilon > 0)\,(\exists (\sigma_n)_{n\in\omega})\left(\sum_{n\in\omega} \frac{1}{2^{|\sigma_n|}} < \varepsilon \wedge F \subseteq \bigcup_{n\in\omega} [\sigma_n]\right).$$

Lemma (Bartoszyński, Judah)

Let $F \subseteq 2^\omega$ be a null set. Then there is a sequence $(S_n)_{n\in\omega}$, $S_n \subseteq 2^n$, $\sum \frac{|S_n|}{2^n} < \infty$, satisfying a condition

$$F \subseteq \{x \in 2^\omega : (\exists^\infty n)(x{\restriction}n \in S_n)\}.$$

Fake measure in the Baire space

Definition

Let $F \subseteq \omega^\omega$. $F \in f\mathcal{N}(h)$ if

$$(\forall \varepsilon > 0)\,(\exists (\sigma_n)_{n\in\omega})\left(\sum_{n\in\omega} \frac{1}{h(|\sigma_n|)} < \varepsilon \wedge F \subseteq \bigcup_{n\in\omega} [\sigma_n]\right).$$

Lemma

Let $F \subseteq \omega^\omega$, $F \in f\mathcal{N}(h)$. Then there is a sequence $(S_n)_{n\in\omega}$, $S_n \subseteq \omega^n$, $\sum \frac{|S_n|}{h(n)} < \infty$, satisfying a condition

$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x{\restriction}n \in S_n)\}.$$

Fake small and closed null sets

Definition

$F \in f\mathcal{E}(h)$ if there are a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $\sum \frac{|J_n|}{h(|I_n|)} < \infty$

$$F \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x \restriction I_n \in J_n)\}.$$

Definition

$F \in f\mathcal{S}(h)$ if there are a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $\sum \frac{|J_n|}{h(|I_n|)} < \infty$

$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x \restriction I_n \in J_n)\}.$$

Observations

- $f\mathcal{N}(h)$ is a translation-invariant $\sigma-$ideal
- $h \leq g \implies f\mathcal{N}(h) \subseteq f\mathcal{N}(g), f\mathcal{E}(h) \subseteq f\mathcal{E}(g), f\mathcal{S}(h) \subseteq f\mathcal{S}(g)$
- $f\mathcal{E}(h) \subsetneq f\mathcal{S}(h)$
- $f\mathcal{N}(h) \perp \mathcal{M}$
- $f\mathcal{N}(h) \not\subseteq \mathcal{K}_\sigma, \mathcal{K}_\sigma \not\subseteq f\mathcal{N}(h)$

Are they surely ideals?

Theorem

Let $h \in \omega^\omega$ be increasing. Then $f\mathcal{E}(h)$ is not an ideal.

Question

Is there a function $h \in \omega^\omega$ such that $f\mathcal{S}(h)$ is an ideal?

Alternative characterization of $f\mathcal{E}(h)$

Theorem

Let $h \in \omega^\omega$ satisfy $h(a+b) \geq h(a)h(b)$ for any $a, b \in \omega$ and $c \in (0,1)$. Then we can replace the condition $\sum_{n\in\omega} \frac{|J_n|}{h(|I_n|)} < \infty$ in the definition of $f\mathcal{E}(h)$ with $(\forall n)(\frac{|J_n|}{h(|I_n|)} < c)$.

Fin-versions

Definition

We will say that

- $F \in f\mathcal{N}(\text{Fin})$ if there is $(S_n)_{n\in\omega}$, $S_n \subseteq \omega^n$, $|S_n| < \omega$ such that
$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x{\restriction}n \in S_n)\}.$$
- $F \in f\mathcal{S}(\text{Fin})$ if there is a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $|J_n| < \omega$, such that
$$F \subseteq \{x \in \omega^\omega : (\exists^\infty n)(x{\restriction}I_n \in J_n)\}.$$
- $F \in f\mathcal{E}(\text{Fin})$ if there is a partition of ω into intervals $(I_n)_{n\in\omega}$ and a sequence $(J_n)_{n\in\omega}$, $J_n \subseteq \omega^{I_n}$, $|J_n| < \omega$, such that
$$F \subseteq \{x \in \omega^\omega : (\forall^\infty n)(x{\restriction}I_n \in J_n)\}.$$

Observations

- $f\mathcal{E}(\mathrm{Fin}) = \mathcal{K}_\sigma \subsetneq f\mathcal{N}(\mathrm{Fin})$
- $f\mathcal{N}(\mathrm{Fin}) \subseteq f\mathcal{S}(\mathrm{Fin})$ and both are $\sigma-$ideals

Chain conditions and cardinal invariants

Theorem

$f\mathcal{N}(\mathrm{Fin})$ *is not* $\mathfrak{c}-cc$.

Theorem

$\mathfrak{b} \leq \mathrm{add}(f\mathcal{N}(\mathrm{Fin}))$ *and* $\mathrm{cof}(f\mathcal{N}(\mathrm{Fin})) \leq \mathfrak{d}$.

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Thank You for attention