$\mathcal{M}^*_-$

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Let $(X, +)$ be an abelian group. For $A, B \subseteq X$ we write

$$A + B = \{a + b : a \in A, b \in B\}.$$

Definition

For a family $\mathcal{F} \subseteq \mathcal{P}(X)$ let:

$$\mathcal{F}^* = \{A \subseteq X : \forall_{F \in \mathcal{F}} A + F \neq X\}.$$

Definition

$M(x, I) = \{y : \forall^\infty n \quad x \restriction I_n \neq y \restriction I_n\}$, where x is some real and I is some interval partition.

Theorem

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Theorem

For every meager set X we have $X \subseteq M(x, I)$ for some $x \in 2^\omega$ and interval partition I.

- We say that for interval partitions I, J we have $I \sqsubseteq^* J$ if for almost every n there exists k such that $I_k \subseteq J_n$.
- If $I \sqsubseteq^* J$, then $M(x, I) \subseteq M(x, J)$.
- If $M(y, I) \subseteq M(z, J)$, then $I \sqsubseteq^* J$.

Theorem

Set $X \subseteq 2^\omega$ has strong measure zero if and only if for every interval partition I there exists $z \in 2^\omega$ such that:

$$\forall x \in X \quad \exists^\infty n \quad x \restriction I_n = z \restriction I_n$$

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Theorem (Galvin-Mycielski-Solovay)

A set $X \subseteq 2^\omega$ has strong measure zero if and only if for every meager set H it holds that $X + H \neq 2^\omega$.

So using *operation we can write

$$\mathcal{SMZ} = \mathcal{M}^*$$

Borel Conjecture

$\mathcal{SMZ} = Count$

Corollary

Borel Conjecture $\Longrightarrow \mathcal{M} \neq \mathcal{M}^{**}$

What about the Baire space?

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Definition

A set $X \subseteq \omega^\omega$ is $\mathcal{M}_-$ if $X \subseteq M(x, I)$ for some interval partition I and $x \in \omega^\omega$

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Set $X \subseteq \omega^\omega$ is $\mathcal{SMZ}^+$ if for every interval partition $I \in \mathbb{IP}$ there exists $z \in \omega^\omega$ such that:

$$\forall x \in X \quad \exists^\infty n \quad x \restriction I_n = z \restriction I_n$$

$$\mathcal{M}_- \subseteq \mathcal{M}$$

$$\mathcal{SMZ} \subseteq \mathcal{SMZ}^+$$

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X$\in \mathcal{SMZ}^+ \iff$ for every $H \in \mathcal{M}_-$ we have $X + H \neq \omega^\omega$

Proof.

($\Longleftarrow$) Assume $X + H \neq \omega^\omega$ for any $H \in \mathcal{M}_-$. Take any partition $I \in \mathbb{IP}$ and take $H \subseteq M(0, I)$. Since $X + H \neq \omega^\omega$, there exists $z \notin X + H \subseteq \bigcup_{x \in X} M(x, I)$. So for every $x \in X$, there exist infinitely many n such that $z \upharpoonright I_n = x \upharpoonright I_n$. So we get:

$$X \subseteq \{x \in \omega^\omega : \exists_n^\infty x \upharpoonright I_n = z \upharpoonright I_n\} \in \mathcal{SMZ}$$

□

Proof.

($\Longrightarrow$) Take any $X \in \mathcal{SMZ}$ and any $H \in \mathcal{M}_-$, where $H \subseteq M(y, I)$ and $X \subseteq \{x : \exists^\infty n \quad x \restriction I_n = z \restriction I_n\}$. We have $X + H \subseteq \bigcup_{x \in X} M(y + x, I)$.

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We have to find $a \notin X + H$. Let $a = y + z$
Assume that $y + z \in X + H$, then for some $x \in X$ and almost all n we have $y + x \restriction I_n \neq y + z \restriction I_n$, so $x \restriction I_n \neq z \restriction I_n$.

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($\Longrightarrow$) Take any $X \in \mathcal{SMZ}$ and any $H \in \mathcal{M}_-$, where $H \subseteq M(y, I)$ and $X \subseteq \{x : \exists^\infty n \quad x \restriction I_n = z \restriction I_n\}$. We have $X + H \subseteq \bigcup_{x \in X} M(y + x, I)$.
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Assume that $y + z \in X + H$, then for some $x \in X$ and almost all n we have $y + x \restriction I_n \neq y + z \restriction I_n$, so $x \restriction I_n \neq z \restriction I_n$.
We get a contradiction, because $X \in \mathcal{SMZ}^+$, so for infinitely many n we have $x \restriction I_n = z \restriction I_n$ $\square$

Cardinal invariants of $\mathcal{M}_-$

$$add(\mathcal{M}_-) = add(\mathcal{M})$$
$$cov(\mathcal{M}_-) = cov(\mathcal{M})$$
$$cof(\mathcal{M}_-) = cof(\mathcal{M})$$
$$non(\mathcal{M}_-) = non(\mathcal{M})$$

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$$cov(\mathcal{M}) = min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \forall y \in \omega^\omega \quad \exists x \in \mathcal{F} \quad x \neq^* y\}$$

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For interval partitions we have:

$$\mathfrak{b} = min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{IP} \wedge \neg\exists J \in \mathbb{IP} \quad \forall I \in \mathcal{F} \quad I \sqsubseteq^* J\}$$

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Proof.

- $add(\mathcal{M}_-) \leq cov(\mathcal{M})$
  Since $cov(\mathcal{M}_-) = cov(\mathcal{M})$ we have that
  $add(\mathcal{M}_-) \leq cov(\mathcal{M}_-) = cov(\mathcal{M})$.

□

Proof.

- $add(\mathcal{M}_-) \leq \mathfrak{b}$
  We want to construct family $\{M_\alpha : \alpha < \mathfrak{b}\} \subseteq \mathcal{M}_-$ such that $\bigcup_\alpha M_\alpha \notin \mathcal{M}_-$.

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  For every M_α let $M_\alpha \subseteq M(x_\alpha, P^\alpha)$ where P^α is unbounded family of interval partitions.

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  For every M_α let $M_\alpha \subseteq M(x_\alpha, P^\alpha)$ where P^α is unbounded family of interval partitions.
  Assume that there exist $X \in \mathcal{M}_-$ such that $\bigcup_\alpha M_\alpha \subseteq X$ and $X \subseteq M(y, Q)$. Then for every α we have $M(x_\alpha, P^\alpha) \subseteq M(y, Q)$.

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  From that for all α we have $P^\alpha \sqsubseteq^* Q$. That is contradiction, because P^α was unbounded.

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Proof.

Assume $add(\mathcal{M}_-) < add(\mathcal{M}) = min\{\mathfrak{b}, cov(\mathcal{M})\}$. Take $(M_\alpha : \alpha < \kappa < add(\mathcal{M}))$ and $M_\alpha \subseteq M(x_\alpha, I^\alpha)$. We will show that $\bigcup M_\alpha \in \mathcal{M}_-$

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Since $\kappa < cov(\mathcal{M})$ there exists $z \notin \bigcup M_\alpha$ so $\forall \alpha \exists_n^\infty \quad z \restriction I_n^\alpha = x_\alpha \restriction I_n^\alpha$.

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Take $(J^\alpha : \alpha < \kappa)$ such that $\forall n \exists k (I^\alpha_k \subseteq J^\alpha_n \wedge z \upharpoonright I^\alpha_k = x_\alpha \upharpoonright I^\alpha_n)$.

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Take $R \in \mathbb{IP}$ such that $\forall \alpha \quad J^\alpha \sqsubseteq^* R$.

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So $\bigcup_{\alpha<\kappa} M_\alpha \subseteq M(z, R)$

□

Moje kočka