Definable Witnesses

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Introduction

This work is at the intersection of:

- Descriptive set theory: studying the properties of definable subsets of the real line.
- Set theory of the reals: studying combinatorial properties of subsets of the real line.

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Motivation

Study the **definability** of combinatorial subsets of reals in various models of set theory, in particular in models with **large continuum** and nontrivial **cardinal characteristic constellations**.

Descriptive set theory

Fix an uncountable Polish space X (e.g., $\mathbb{R}$, $[0,1]$, ω^ω, $[\omega]^\omega \dots$). The Borel ($\mathbf{\Delta}^1_1$) subsets of X is the σ-algebra generated by the open ($\mathbf{\Sigma}^0_1$) subsets of X.

$$\begin{array}{ccccccccccccccc}
 & \mathbf{\Sigma}^0_1 & & \mathbf{\Sigma}^0_2 & & \mathbf{\Sigma}^0_3 & & & & \mathbf{\Sigma}^1_1 & & \mathbf{\Sigma}^1_2 & & \mathbf{\Sigma}^1_3 & \\
\mathbf{\Delta}^0_1 & & \mathbf{\Delta}^0_2 & & \mathbf{\Delta}^0_3 & & \cdots & & \mathbf{\Delta}^1_1 & & \mathbf{\Delta}^1_2 & & \mathbf{\Delta}^1_3 & & \cdots \\
 & \mathbf{\Pi}^0_1 & & \mathbf{\Pi}^0_2 & & \mathbf{\Pi}^0_3 & & & & \mathbf{\Pi}^1_1 & & \mathbf{\Pi}^1_2 & & \mathbf{\Pi}^1_3 &
\end{array}$$

Borel hierarchy — projective hierarchy

The projective hierarchy

$\mathbf{\Sigma}^1_1$ = analytic sets (projections of $\mathbf{\Pi}^0_1$ subsets of $X \times \omega^\omega$)

$\mathbf{\Sigma}^1_{n+1}$ = projections of $\mathbf{\Pi}^1_n$ sets

$\mathbf{\Pi}^1_1$ = coanalytic sets (complements of $\mathbf{\Sigma}^1_1$ sets)

$\mathbf{\Pi}^1_{n+1}$ = complements of $\mathbf{\Sigma}^1_{n+1}$ sets

$$\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$$

The constructible universe L and its definable combinatorics

Theorem (Gödel [Göd39])

If $V = L$ then there exists a Δ^1_2-definable wellorder of the reals.

- V=L can be used to produce combinatorially interesting sets with minimal descriptive complexity: an *optimal projective witness*.

Corollary (Gödel [Göd39])

It is consistent that there exists a Δ^1_2 set which is not Lebesgue measurable nor has the property of Baire, and there exists a Π^1_1 set without the perfect set property.

Forcing and large continuum

Theorem (Harrington, 1977 [Har77])

It is consistent that $\mathfrak{c}$ is arbitrarily large and:

- *there exists a $\mathbf{\Delta}^1_3$ wellorder of the reals;*
- *there exists a $\mathbf{\Pi}^1_2$ wellordering of a set of reals of length $\mathfrak{c}$;*
- *MA holds.*

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This is optimal:

Theorem (Mansfield)

If R is a Σ^1_2 wellordering of a set of reals then R is of length $\leq \aleph_1$. Moreover, if there exists a Σ^1_2 total wellorder of the reals, then $\mathbb{R} \subseteq L$.

Our work

V=L

Δ^1_2 wellorder, nice definable combinatorics; all interesting combinatorial sets of size $\mathfrak{c} = \aleph_1$.

Forcing extensions

Projective wellorders with $\neg$CH; interesting combinatorial cardinal characteristic phenomena.

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Forcing extensions

Projective wellorders with $\neg$CH; interesting combinatorial cardinal characteristic phenomena.

Questions

1. How can we obtain optimal definable witnesses as done in L, but in models of $\neg$CH?
2. With which cardinal characteristic constellations is an $\aleph_1$-sized definable witness compatible?
3. How can we obtain definable witnesses of size strictly greater than $\aleph_1$?

Coanalytic witnesses in L

- Objects constructed from a wellordering of the reals are typically not Δ^1_1- or Σ^1_1-definable.

$A \subseteq X$ is Σ^1_2 $\Rightarrow$ there is $F \subseteq X \times \omega^\omega$ such that

$$x \in A \Leftrightarrow \exists y\ (x, y) \in F.$$

- Robust coding methods under $V = L$ to produce Π^1_1 witnesses.
  - Originates in Erdős, Kunen, and Mauldin [EKM81];
  - Streamlined by Arnold Miller, numerous applications [Mil89];
  - Formalized by Vidnyánszky, systematized machinery [Vid14].

Question

What about their arguments can be extracted and applied to models of $\neg$CH?

Coanalytic witnesses from Σ^1_2 sets

Theorem (Mathias, 1969; [Mat77])

If $\mathcal{A}$ is an analytic almost disjoint family, then $\mathcal{A}$ is not maximal.

Theorem (Miller, 1989; [Mil89])

If $V = L$ then there exists a Π^1_1 mad family.

Theorem (Törnquist, 2013; [T09])

If there exists a Σ^1_2 mad family, then there exists a Π^1_1 mad family.

Coanalytic witnesses from Σ^1_2 sets

Definition

A mad family $\mathcal{A} \in V$ is $\mathbb{P}$-*indestructible* for some forcing notion $\mathbb{P}$ if $\mathcal{A}$ remains maximal in any $\mathbb{P}$-generic extension over V.

General strategy:

- Construct a $\mathbb{P}$-indestructible mad family in L in a Σ^1_2-way, where $\mathbb{P}$ is some forcing adding reals.
- In $L^{\mathbb{P}}$, $\mathcal{A}$ is a Σ^1_2 mad family, so apply Törnquist's theorem.

Corollary

- *It is consistent with $\mathfrak{a} < \mathfrak{c}$ that there exists a coanalytic witness of size $\mathfrak{a} = \aleph_1$.*
- *(Brendle, Khomskii [BK13]) It is consistent that $\aleph_1 < \mathfrak{a} = \mathfrak{c} = \kappa$ regular cardinal, and there exists a coanalytic mad family.*

A general framework for removing an existential quantifier

What we do:

- Survey the literature for implications of the form $\Sigma^1_2 \Rightarrow \Pi^1_1$.
  - Maximal independent families, towers, maximal eventually different families of functions, maximal orthogonal families, ultrafilter bases.
- Extract a general framework; present the theorems in a uniform fashion.
  - Essential use of Π^1_1-uniformization, Spector-Gandy theorem.
- Apply the framework to produce a new such theorem: Hausdorff gaps in $(\mathcal{P}(\omega), \subseteq^*)$.

See also [Mil24].

The framework is potent: has been applied to several other combinatorial structures (M., Schembecker, 2025 [MS25]).

Hausdorff gaps

Definition

A *pre-gap* is a pair (A, B) where $A, B \subseteq [\omega]^\omega$ are wellordered by the relation $\subseteq^*$, and for all $a \in A$ and $b \in B$, $a \cap b$ is finite.[1]. A pre-gap (A, B) is a *gap* if there does not exist $c \in [\omega]^\omega$ such that for all $a \in A$ and $b \in B$, $a \cap c$ is finite and $c \subseteq b$. (Such a c *separates* A, B).

- Found in work of Hausdorff, Du Bois-Reymond, Hadamard.

Theorem (Todorcevic, 1996; [Tod96])

Suppose (A, B) is a pre-gap and A is analytic. Then (A, B) can be separated.

[1] We define $x \subseteq^* y$ if and only if $x \setminus y$ is finite

Hausdorff Gaps

Definition

A *Hausdorff gap* is a pre-gap (A, B), where $A = \{a_\alpha \mid \alpha < \omega_1\}$, $B = \{b_\alpha \mid \alpha < \omega_1\}$, satisfying:

$$\forall \alpha < \omega_1 \forall k < \omega [\{\gamma < \alpha \mid a_\alpha \cap b_\gamma \subseteq k\}] \text{ is finite.}$$

(A, B) is a Hausdorff gap $\Rightarrow$ (A, B) is a gap.

Theorem (M., 2024)

If there exists a Hausdorff gap (A, B) such that both A, B are Σ^1_2-definable, then there exists a Hausdorff gap (A', B') such that A', B' are Π^1_1-definable.

Corollary

It is consistent with $\mathfrak{c} > \aleph_1$ that there exists a Hausdorff gap (A, B), and (A, B) are coanalytic.

Cardinal characteristics and definable spectra

- Can we obtain Π^1_1 small witnesses while also controlling the values of other cardinal characteristics?
- What about the combinatorial sets of size $> \aleph_1$?

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- How can we control which cardinals κ belong to $\mathrm{spec}(\mathfrak{a}) = \{|\mathcal{A}| \mid \mathcal{A} \text{ is mad}\}$ *and* ensure that for each such κ there is a projective mad family with an optimal definition?

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Known

A definable witness to the value of $\mathfrak{a}$ is consistent with:

- (Friedman, Zdomskyy, [FZ10]) $\mathfrak{b} = \mathfrak{c} = \aleph_2$
- (Fischer, Friedman, Zdomskyy, [FFZ11]; Brendle, Khomskii ([**?**]) $\mathfrak{b} = \mathfrak{c} = \kappa \geq \aleph_3$;
- (Fischer, Friedman, Schrittesser, Törnquist [FFST25]) $\aleph_1 < \mathfrak{a} < \mathfrak{c}$.

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Known

- Hechler pioneered the study of $\mathrm{spec}(\mathfrak{a})$; pursued by Blass [Bla10], Shelah and Spinas [SS15].
- Substantial control of realizing $\mathrm{spec}(\mathfrak{a}) = C$ for a given subset of cardinals C.

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Theorem (Fischer, M., 2025)

It is consistent with $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists:

- *A Δ^1_3 wellorder of the reals;*
- *A coanalytic tight mad family of size $\aleph_1$;*
- *A Π^1_2 tight mad family of size $\aleph_2$.*

Proof strategy

Theorem (Fischer, M., 2025)

It is consistent with $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists:

- *A Δ^1_3 wellorder of the reals;*
- *A coanalytic tight mad family of size $\aleph_1$;*
- *A Π^1_2 tight mad family of size $\aleph_2$.*

Fixing a Π^1_1 tight mad family $\mathcal{A}$ in L, there are three tasks:

1. Increase the size of $\mathfrak{s}$;
2. Add a Δ^1_3 wellorder of the reals;
3. Add a Π^1_2 tight mad family of size $\mathfrak{c}$.

Crucially, we ensure this can be done without destroying the maximality of $\mathcal{A}$, so cannot add dominating reals.

Tight mad families

Definition

- For an almost disjoint family $\mathcal{A}$, the *ideal generated by* $\mathcal{A}$ is the set

$$\mathcal{I}(\mathcal{A}) := \{b \in [\omega]^\omega \mid b \subseteq^* \bigcup F \text{ for some finite } F \subseteq \mathcal{A}\}.$$

- An almost disjoint family $\mathcal{A}$ is *tight* if for all countable $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})^+ = [\omega]^\omega \setminus \mathcal{I}(\mathcal{A})$, there exists a single $a \in \mathcal{A}$ such that $a \cap b$ is infinite for each $b \in \mathcal{B}$.

$\mathcal{A}$ is tight $\Rightarrow$ $\mathcal{A}$ is maximal.

- Exist under CH and $\mathfrak{b} = \mathfrak{c}$. Their existence under ZFC is a long standing open problem.
- Introduced by Malykhin [Mal89]; are Cohen-indestructible [Kur01].

Preservation of tight mad families

- 2020: Guzman, Hrusak, and Tellez [GHT20] introduce a preservation notion for tight mad families: *strong preservation of tightness*.
  Crucially:

Theorem (Guzman, Hrušák, Tellez, 2020; [GHT20])

If $\mathbb{P}$ is a countable support iteration of length $\delta \leq \omega_2$ of forcings which strongly preserve the tightness of $\mathcal{A}$, then $\mathbb{P}$ strongly preserves the tightness of $\mathcal{A}$.

- Examples: Sacks forcing, Miller forcing, Miller partition forcing.

Creature forcing

Task 1

Increase $\mathfrak{s}$, but do not increase $\mathfrak{a}$.

ZFC proves the following:

$$\begin{array}{ccccc} \mathfrak{s} & \longrightarrow & \mathfrak{d} & \longrightarrow & \mathfrak{c} \\ \uparrow & & \uparrow & & \uparrow \\ \aleph_1 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{a} \end{array}$$

- (Balcar, Simon [BPS80]) Consistency of $\mathfrak{s} < \mathfrak{b}$; also holds in Hechler model.
- $\mathfrak{d}$ and $\mathfrak{a}$ are independent.
- Shelah [She84] introduces first *creature forcing* to show $\mathfrak{b} < \mathfrak{s}$ is consistent.

Creature forcing

Task 1

Increase $\mathfrak{s}$, but do not increase $\mathfrak{a}$.

- Mathias forcing: increases $\mathfrak{s}$ by adding an unsplit real, but also increases $\mathfrak{b}$, since it adds a dominating real.
- Shelah's creature forcing $\mathbb{Q}$ of [She84] increases $\mathfrak{s}$, but is *almost ω^ω-bounding*, meaning $\mathbb{Q}$ (and its iterations) keep $\mathfrak{b}$ small.
- almost ω^ω-bounding $\not\Rightarrow$ preserves mad families.
- Original proof of the consistency of $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ was by directly constructing a mad family in the ground model.
- Alan Dow [Dow95] uses a similar approach to show $\mathfrak{a} = \aleph_1$ in the Miller model.

A creature forcing preserves tightness

Proposition (Fischer, M., 2025)

Let $\mathbb{Q}$ be Shelah's creature forcing [She84], let $\mathcal{A} \in V$ be a tight mad family. Then $\mathbb{Q}$ strongly preserves the tightness of $\mathcal{A}$.

This reveals that $\mathbb{Q}$ has a stronger combinatorial property than almost ω^ω-bounding, and it follows:

Theorem (Shelah, [She84])

Assume CH, and let $\mathbb{P}$ be an ω_2-length countable support iteration of $\mathbb{Q}$, and let G be $\mathbb{P}$-generic over V. Then $V[G] \models \aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$.

Therefore, also in this model $\mathfrak{b} < \mathfrak{s}$.

Projective wellorders and cardinal characteristics

Task 2

Add a Δ^1_3 wellordering of the reals, while preserving a tight mad family in the ground model.

The existence of definable wellorders is a central question in set theory. Projective wellorderings of the reals indicate to what extent regularity properties hold for the projective classes.

- $V = L \Rightarrow$ there exists a Δ^1_2 wellorder of the reals.
- (Harrington, 1977) A Δ^1_3 wellorder of the reals is consistent with $\mathfrak{c}$ arbitrarily large, and Martin's Axiom.
- (Caicedo, Friedman, 2011 [CF11]) Under the Bounded Proper Forcing Axiom and an anti-large cardinal hypothesis, there exists a Δ^1_3 wellorder of the reals.

Projective wellorders and cardinal characteristics

Theorem (Fischer, Friedman, 2010 [FF10])

Each of the following cardinal inequalities is consistent with $\mathfrak{c} = \aleph_2$ and the existence of a Δ^1_3 wellordering of the reals: $\mathfrak{d} < \mathfrak{c}$, $\mathfrak{b} < \mathfrak{g}$, and $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$.

- Countable support iteration of S-proper forcings, where $S \in L$ is a fixed stationary subset of ω_1.
- Generic reals originate from *Sacks coding*: this gives a way of coding with reals via an ω^ω-bounding forcing.
- Coding apparatus: club shooting patterns through an ω_2-length sequence of stationary costationary sets from the ground model; David's trick [Dav82].
- "Flexible": any proper forcing notion of size at most $\aleph_1$ may be woven in.

Projective wellorders and tight mad families

Lemma (Bergfalk, Fischer, Switzer, 2022; [BFB22])

The Sacks coding forcing strongly preserves the tightness of ground model coanalytic tight mad families $\mathcal{A}$ such that ZFC proves $\mathcal{A} \subseteq L$.

By weaving in the creature forcing to the Fischer-Friedman construction:

Theorem (Fischer, M., 2025)

It is consistent that there is a Δ^1_3 wellorder of the reals, $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$, and there is a coanalytic mad family of size $\aleph_1$.

The Friedman-Zdomskyy Forcing

Task 3

Add a Π^1_2 mad family of size $\mathfrak{c} = \aleph_2$, while preserving a tight mad family in the ground model.

Theorem (Friedman, Zdomskyy (2010) [FZ10])

It is consistent with $\mathfrak{b} = \mathfrak{c} = \aleph_2$ that there exists a Π^1_2 tight mad family.

- Countable support iteration of S-proper forcings.
- Appropriate bookkeeping to take care of all $\mathcal{B} \in [\mathcal{I}(\mathcal{A}_\alpha)^+]^\omega$, where $\mathcal{A}_\alpha$ is the family constructed up to stage α.
- Generic reals arise from *almost-disjoint coding*, a technique of Solovay and Jensen [SJ70], crucial to Harrington's work.
- [FZ10] also adds ω_2-many Hechler reals to ensure $\mathfrak{a} = \mathfrak{c}$.

Definability of tight mad families

Theorem (Raghavan, [Rag09])

If $\mathcal{A}$ is a tight mad family, then $\mathcal{A}$ does not contain a perfect set.

Theorem (Mansfield, Solovay, 1950)

Suppose A is a Σ^1_2 set of reals such that there exists $a \in A$ with $a \notin L$. Then A contains a perfect set of nonconstructible reals.

Hence, the minimal complexity of a projective tight mad family of size $\geq \aleph_2$ is Π^1_2.

A preservation theorem

Proposition

Let $\mathbb{K}$ be the Friedman-Zdomskyy forcing for adding a Π^1_2 tight mad family of size $\mathfrak{c}$. Then $\mathbb{K}$ strongly preserves the tightness of ground model tight mad families.

Thus, $\mathbb{K}$ above does not add dominating reals.

Theorem (Fischer, M., 2025)

It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{c} = \aleph_2$ that there exists a coanalytic tight mad family of size $\aleph_1$ and a Π^1_2 tight mad family of size $\aleph_2$.

Orchestrating a symphony

Theorem (Fischer, M., 2025, [FM25])

It is consistent with $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists:

- *A Δ^1_3 wellorder of the reals;*
- *A coanalytic tight mad family of size $\aleph_1$;*
- *A Π^1_2 tight mad family of size $\aleph_2$.*

Proof.

Beginning in a model of $V = L$, use the template for adding a Δ^1_3 wellorder, weaving in either:

1. Shelah's forcing $\mathbb{Q}$;
2. The Friedman-Zdomskyy forcing $\mathbb{K}$.

(1)$\Rightarrow$ $\mathfrak{a} = \aleph_1 < \mathfrak{s} = \aleph_2$
(2)$\Rightarrow$ there exists Π^1_2 tight mad family of size $\mathfrak{c}$

□

Closing remarks and open questions

Remark

Some combinatorial families can be analytic, even Borel. This is the case for maximal eventually different families and maximal cofinitary groups ([HS24], [HS25]). Such Borel witnesses are always of size $\mathfrak{c}$.

Theorem

- *(Fischer, Schrittesser; [FS21]) It is consistent with* $\mathfrak{a}_e = \aleph_1 < \mathfrak{c}$ *that there exists a coanalytic maximal eventually different family of size* $\aleph_1$
- *(Fischer, Schrittesser, Törnquist; [FST17]) It is consistent with* $\mathfrak{a}_g = \aleph_1 < \mathfrak{c}$ *that there exists a coanalytic maximal cofinitary group of size* $\aleph_1$*.*

Closing remarks and open questions

Open questions:

1. Can the proofs of the form "if there exists a Σ^1_2 family of type P then there exists a Π^1_1 family of type P" be lifted to the levels Σ^1_{n+1} and Π^1_n, $n \geq 2$?
   - Joint work with Fischer, Khomskii, Li.
2. Can we have a model with $\mathrm{spec}(\mathfrak{a}) > 2$ and for every $\kappa \in \mathrm{spec}(\mathfrak{a})$ there exists a mad family of size κ with an optimal projective definition?
   - Not known how to obtain the above while also demanding $\aleph_1 \in \mathrm{spec}(\mathfrak{a})$.

The end

Thank you! Dziekuje! Merci! Vielen Dank! Arigato! Gracias! Grazi! Dank je! Hvala! Ďakujem! Kiitos! Köszönöm!

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