A 2 generator free LD-algebra of embeddings

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Joint work with Scott Cramer and Sheila Miller Edwards
with improvements thanks to Gabe Goldberg

Winter School
1 February 2026

Large cardinal axioms

Many have the form:

> *there is a non-trivial elementary embedding from the universe of sets V to a submodel M with suitable closure*

i.e. there is $j\colon V \to M$ such that for every formula φ in the language of set theory and all parameters $a_1, \ldots, a_n$,

$$(V, \in) \vDash \varphi(a_1, \ldots, a_n) \qquad \text{if and only if} \qquad (M, \in) \vDash \varphi(j(a_1), \ldots, j(a_n)).$$

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"The" large cardinal is then the *critical point* of j,

$$\mathrm{crit}(j) = \min(\{\alpha \mid j(\alpha) \neq \alpha\}).$$

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- All rely heavily on the Axiom of Choice.
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We're interested in a different way to step back from inconsistency.

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So let's just stay below $\lambda + 2$ to avoid the inconsistency!

Axiom I3: $\exists$ elementary $j\colon V_\lambda \to V_\lambda$

Axiom I1: $\exists$ elementary $j\colon V_{\lambda+1} \to V_{\lambda+1}$

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- Something even stronger than I1 was used by Woodin in his original work on the consistency of AD (before bringing it way down).
- 50 years of working with them hasn't turned up any more inconsistencies.

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For *any* $j \in \mathcal{E}_\lambda$, we can treat it as a function from $V_{\lambda+1}$ to $V_{\lambda+1}$ with this definition of $j(A)$ for $A \subseteq V_\lambda$, even though it won't in general be elementary as a function from $V_{\lambda+1}$ to $V_{\lambda+1}$.

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We say that $j \colon V_\lambda \to V_\lambda$ is Σ^1_m-elementary if for all Σ^1_m formulas φ and parameters $a_1, \ldots, a_{n_0}, A_1, \ldots, A_{n_1}$,

$$V_\lambda \vDash \varphi(a_1, \ldots, a_{n_0}, A_1, \ldots, A_{n_1})$$

if and only if

$$V_\lambda \vDash \varphi(j(a_1), \ldots, j(a_{n_0}), j(A_1), \ldots, j(A_{n_1}))$$

This gives another standard large cardinal axiom:

I2: $\exists\ \Sigma^1_1$-elementary $j\colon V_\lambda \to V_\lambda$.

We'll come back to this.

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Lemma

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Proof outline.

Elementarity of k can be expressed by: for every formula φ and $\alpha < \lambda$,

$$V_\lambda \models \forall a_1, \ldots, a_n \in V_\alpha(\varphi(a_1, \ldots, a_n) \leftrightarrow \varphi((k \upharpoonright V_\alpha)(a_1), \ldots, (k \upharpoonright V_\alpha)(a_n)).$$

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By elementarity of j we then have

$$V_\lambda \vDash \forall a_1, \ldots, a_n \in V_{j(\alpha)}(\varphi(a_1, \ldots, a_n) \leftrightarrow \varphi(j(k \restriction V_\alpha)(a_1), \ldots, j(k \restriction V_\alpha)(a_n)).$$

□

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Note

$j * k$ is very different from $j \circ k$. E.g.,

$$\mathrm{crit}(j * k) = j(\mathrm{crit}(k)), \qquad \text{but} \qquad \mathrm{crit}(j \circ k) = \min(\mathrm{crit}(j), \mathrm{crit}(k)).$$

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For any $j, k, \ell \in \mathcal{E}_\lambda$, we have by elementarity that

$$j * (k * \ell) = (j * k) * (j * \ell).$$

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Definition

A set with a binary operation $*$ satisfying $a * (b * c) = (a * b) * (a * c)$ is an *LD-algebra* or *shelf*.

Let $j \colon V_\lambda \to V_\lambda$ be a rank-to-rank embedding and let $\mathcal{A}_j$ be the structure consisting of embeddings generated by j under the $*$ operation.

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Theorem (Laver)

$\mathcal{A}_j$ is the free LD-algebra on the single generator j.

Some elements of $\mathcal{A}_j$

Definition

For $n \in \omega$ we define recursively $j^{(n)}$, the *nth right power of* j, by

$$j^{(0)} = j$$
$$j^{(n+1)} = j * j^{(n)}$$

So $j^{(n)} = j(j(\cdots(j(j))\cdots))$ with $n+1$ "j"s.

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By induction using distributivity, for all n we have

$$j^{(n+1)} = j^{(n)} j^{(n)},$$

expanding all the way out to an expression with 2^n "j"s.

Also. . .

Critical points

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Note: not every critical point of an embedding in $\mathcal{E}_\lambda$, or even in $\mathcal{A}_j$, is of this form. For example

$$\kappa_2 < \mathrm{crit}(((jj)j)(jj)) < \kappa_3!$$

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Inverse limits

If $(k_n)_{n\in\omega}$ is a sequence of elements of $\mathcal{E}_\lambda$ with critical points increasing with supremum λ, then $k\colon V_\lambda \to V_\lambda$ defined by

$$k(a) = k_0 \circ k_1 \circ \cdots \circ k_m(a),$$

where m is sufficiently large that $k_p(a) = a$ for all $p > m$, is called the *inverse limit* of the k_n.

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It turns out (see Laver) that this is also in $\mathcal{E}_\lambda$, and if all the k_n are Σ^1_m-elementary, so is k.

So given there are so many embeddings in $\mathcal{E}_\lambda \setminus \mathcal{A}_j$, can we push Laver's freeness result further?

Question

Can we find two embeddings k and ℓ in $\mathcal{E}_\lambda$ such that the algebra $\mathcal{A}_{k,\ell}$ generated by k and ℓ is the free LD-algebra on 2 generators?

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Answer (B-T, Cramer, Miller Edwards)

Assuming I2, yes!

Theorem

Suppose there is a Σ^1_1-elementary embedding $j\colon V_\lambda \to V_\lambda$. Then in $\mathcal{E}_\lambda$ there are embeddings k and ℓ such that the LD-subalgebra $\mathcal{A}_{k,\ell}$ of $\mathcal{E}_\lambda$ generated by k and ℓ under $*$ is free on the two generators k and ℓ.

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Improvement (Goldberg)

In fact, I3 suffices.

Proof sketch

We use the embeddings

$$k = j \circ j^{(1)} \circ j^{(2)} \circ j^{(3)} \circ \cdots$$

and

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Central idea: push this harder, making their "iterated ranges" disjoint, to make the choice of k or ℓ detectable from what the embedding does to ordinals.

A tool from algebra

Theorem (Dehornoy)

For two words u and v in the free 2 generator LD-algebra (built as a term algebra) we have a quadrichotomy: one of the following holds.

- *They are equivalent.*
- *u is equivalent to a left subterm of v.*
- *v is equivalent to a left subterm of u.*
- *They have a variable clash: they have a common left-most component, followed by a different choice between the two generators, possibly followed by more.*

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Having strongly disjoint iterated ranges of the generators k and ℓ means that in the variable clash case, the words u and v yield different embeddings.

How do we get that the ranges of k and ℓ are disjoint in a sufficiently strong sense to be able to detect a variable clash?

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By assuming j is Σ^1_1 elementary, we get that it has *square roots*, and then there is enough elementarity floating around to in fact give that

$$[\kappa_{2n}, \kappa_{2n+1}) \ \cap \ \mathit{Def}^{V_\lambda}(\mathrm{rng}(k) \cup V_{\kappa_{2n}}) = \emptyset \tag{$\dagger$}$$

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Observation (Goldberg): actually, j an I3 embedding suffices to get (†) and (‡). □

Question: what about more than 2 generators?

Answer:

Sheila and I are pretty sure we can use the same methods to get an LD-subalgebra of $\mathcal{E}_\lambda$ that is free with continuum many generators (but we're still writing it up).