New results and problems related with Δ-spaces and Δ_1-spaces

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7. (MA) $\wedge(\sim$ (CH)): $X \subset \mathbb{R}$, $|X| < \mathfrak{c}$, then X is a Q-set (Martin-Solovay (1970), M.E. Rudin (1977)).

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Theorem 2 (J.K-Leiderman PAMS (2021, 2022))

(1) Every metrizable σ-scattered space is a Δ-space.
(2) Compact Δ-spaces are scattered, $[0, \omega_1]$ is not a Δ-space.
(3) Compact Eberlein X is a Δ-space iff X is scattered.
(4) No Δ-space with a countable network has cardinality $\mathfrak{c}$.
*(5) X is a Δ-space iff $C_p(X)$ is **distinguished**, i.e. the strong topological dual of $C_p(X)$ carries the finest locally convex topology.*

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A Banach space E is an **Asplund space** if for every convex continuous function f on E the set of all points of Fréchet differentiability of f is **dense and G_δ in** E.

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On ω-resolvable Δ-spaces and Δ_1-spaces

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1. X is *crowded*, if X does not admit isolated points; X is *resolvable* (ω-*resolvable*), if it can be partitioned into 2 (countably many) dense subsets (Hewitt, Ceder). Arc connected crowded spaces are ω-resolvable, so every TVS over $\mathbb{R}$ is ω-resolvable.

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6. (V=L) $\Rightarrow$ (AC) $\wedge$ (GCH).

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3. If X has an isolated point then, trivially, the intersection of any family of dense subsets of X is non-empty.

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Every pseudocompact Δ-space X with $c(X) = \omega$ is separable and has a dense set of isolated points.

3. Consider some cases for which Problem 8 has a positive answer.

1. Under Souslin hypothesis (SH) (i.e. there are no Souslin lines) if X is crowded and Baire with $c(X) = \omega$, then X is ω-resolvable (Casarrubias-Segura, Hernandez-Hernandez, Tamaris-Mascar) (Recall that $(MA) \wedge (\sim (CH)) \Rightarrow (SH)$.)

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(SH) If X is a Baire space with $c(X) = \omega$ and X is a Δ-space, then X has isolated points.

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2. The axiom of constructibility, $V = L$, implies that every crowded Baire space is ω-resolvable (Pavlov). Hence, applying Leiderman-Szeptycki theorem, (under $V = L$) **every Δ-space which is Baire has isolated points**.

1. (under (MA)) (Casarrubias-Segura, Tamaris-Mascar): *Crowded X is ω-resolvable if X is Baire and $c(X) = \omega$* or *X contains a π-network $|\mathcal{U}| < \mathfrak{c}$ of infinite sets.*

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3. Cardinal number κ is measurable if there exists a κ-additive,nontrivial, $0-1$-valued measure on 2^{κ}.

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There exist crowded ω-resolvable countably compact Δ_1-spaces which are not Δ-spaces.

2. Example 13 shows that extending Proposition 7 (due to Leiderman-Szeptycki) to Δ_1-spaces requires some extra assumption on Baire X.

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2. Dealing with Δ_1-spaces we propose the following extension of Proposition 7 as follows.

Theorem 15 (J.K.-Leiderman-Tkachuk)

If X is a separable crowded Baire space, then X is not a Δ_1-space. Hence, a separable Baire space which is a Δ_1-space has isolated points.

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Proof.

Juhasz and van Mill showed that there exists a dense countably compact subspace $Z \subset \beta\omega \setminus \omega$ in which all countable subsets are scattered and hence Z is a Δ_1-space. If $X = \omega \cup Z$, then X is a countably compact separable Δ_1-space not scattered. □

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3. Recall compact X is scattered iff X is a Δ_1-space.

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Problem 19 (Leiderman-Tkachuk)

Suppose that X is a space and there is a family $(U_n)_n$ of open Δ-subspaces of X covering X. Is it true that X must be a Δ-space?

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If X is a Δ-space and $cf(o(X)) > \omega$, then $|X| < o(X)$, where $o(X)$ means the number of all open sets in X.

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Does every hereditary Lindelöf Δ-space have cardinality $< c$?