Cardinal invariants of idealized Miller null sets

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TECHNISCHE UNIVERSITÄT WIEN

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Ideals

We say that $\mathcal{I} \subseteq \wp(X)$ is an ideal on a set X if for all $A, B \subseteq X$,

1. $A \in \mathcal{I} \land B \subseteq A \Rightarrow B \in \mathcal{I}$ and
2. $A \in \mathcal{I} \land B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$.

Furthermore, if $\mathcal{I}$ is closed under countable unions, $\mathcal{I}$ is called a σ-ideal.

Our convention:

- $\mathscr{I}$ denotes an ideal on a countable set X.
- We always assume that $\mathscr{I}$ is proper, i.e. $[X]^{<\omega} \subseteq \mathscr{I}$ and $X \notin \mathscr{I}$.

There are various cardinal invariants associated with a σ-ideal $\mathcal{I}$ on a Polish space X such as:

- Uniformity number: $\mathrm{non}(\mathcal{I}) = \min\{A \subseteq X : A \notin \mathcal{I}\}$.
- Covering number: $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \bigcup \mathcal{F} = X\}$.
- Additivity number: $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \bigcup \mathcal{F} \notin \mathcal{I}\}$.
- Cofinality number: $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \forall A \in \mathcal{I}\, \exists B \in \mathcal{F}\, (A \subseteq B)\}$.

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$\mathscr{I}$-Miller null ideal

Let $\mathscr{I}$ be an ideal on a countable set X.

Definition

The $\mathscr{I}$-Miller null ideal $M_{\mathscr{I}}$ is the σ-ideal on X^ω generated by sets of the form

$$M_\phi = \{f \in X^\omega : \forall^\infty n < \omega \ f(n) \in \phi(f \restriction n)\},$$

where $\phi\colon X^{<\omega} \to \mathscr{I}$.

We also consider the following variant.

Definition

The ideal $K_{\mathscr{I}}$ is the σ-ideal on X^ω generated by sets of the form

$$K_\phi = \{f \in X^\omega : \forall^\infty n < \omega \ f(n) \in \phi(n)\},$$

where $\phi\colon X^{<\omega} \to \mathscr{I}$.

Note that $M_{\mathrm{Fin}} = K_{\mathrm{Fin}} = K_\sigma$, where Fin is the ideal of finite subsets of ω. Also,

$$K_\sigma \subseteq K_{\mathscr{I}} \subseteq M_{\mathscr{I}} \subseteq \mathcal{M}.$$

A natural guiding question would be: When is $\mathrm{non}(M_{\mathscr{I}})$ different from $\mathfrak{b}$ and $\mathrm{non}(\mathcal{M})$?

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Why should $M_{\mathscr{I}}$ be called the $\mathscr{I}$-Miller null ideal?

Let $\mathscr{I}$ be an ideal on a countable set X.

Definition

A tree $T \subseteq X^{<\omega}$ is an $\mathscr{I}$-Miller tree if for every $\sigma \in T$ there exists $\tau \in T$ with $\sigma \subseteq \tau$ and

$$\mathrm{succ}_T(\tau) := \{x \in X : \tau^\frown\langle x\rangle \in T\} \in \mathscr{I}^+.$$

Let $\mathbb{M}_{\mathscr{I}}$ be the forcing poset of $\mathscr{I}$-Miller trees ordered by inclusion. The original Miller forcing is $\mathbb{M}_{\mathrm{Fin}}$.

Sabok–Zapletal showed that $\mathbb{M}_{\mathscr{I}}$ is forcing equivalent to the poset $\mathbb{P}_{M_{\mathscr{I}}}$ of $M_{\mathscr{I}}$-positive Borel sets ordered by inclusion:

Proposition (Sabok–Zapletal)

For every analytic subset $A \subseteq X^\omega$, either $A \in M_{\mathscr{I}}$ or there is an $\mathscr{I}$-Miller tree $T \subseteq X^{<\omega}$ such that all the infinite branches through T are in A.

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$M_{\mathscr{I}}$ and $K_{\mathscr{I}}$ in previous literature

The systematic study of $M_{\mathscr{I}}$ was initiated by Cieślak and Martínez-Celis. However, Brendle and Shelah did a similar study for Laver and Mathias type forcings associated with ultrafilters. Moreover, the cardinal invariants of $M_{\mathscr{I}}$ and $K_{\mathscr{I}}$ were (implicitly) studied from different perspectives.

1. **Forcing perspective**
   - (Sabok–Zapletal) The $\mathscr{I}$-Miller forcing $\mathbb{M}_{\mathscr{I}}$ was studied.
   - (Pawlikowski) A connection between perfect sets of random reals and $K_{\mathcal{Z}}$ was shown: If there is a perfect set of random reals, $\omega^\omega \cap V \in K_{\mathcal{Z}}$.
2. **Topological perspective**
   - (Šupina) $\mathrm{cov}(K_{\mathscr{I}})$ is the least size of non-$S_1(\mathscr{I}\text{-}\Gamma, \mathcal{O})$ Hausdorff spaces.
   - (Cardona–Gavaloá–Mejía–Repický–Šupina) $\mathrm{non}(K_{\mathscr{I}})$ and $\mathrm{cov}(K_{\mathscr{I}})$ appear as slalom numbers.
3. **Combinatorial perspective**
   - (Blass) Uniformity/covering numbers of $M_{\mathscr{I}}/K_{\mathscr{I}}$ can be seen as evasion/prediction numbers (e.g. $\mathrm{non}(M_{\mathscr{I}})$ is the evasion number for global adaptive predictors with values in $\mathscr{I}$).

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Katětov order $\leqslant_K$ is one of the main tools for classification of ideals. For given ideals $\mathscr{I}$, $\mathscr{J}$ on X, Y respectively, define

$$\mathscr{I} \leqslant_K \mathscr{J} \iff \exists f\colon Y \to X\, \forall A \in \mathscr{I}\, (f^{-1}[A] \in \mathscr{J}).$$

The below diagram shows Katětov order on various ideals on countable sets.

Katětov order on ideals on a countable set

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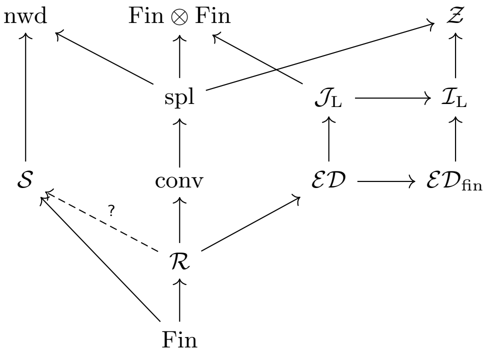

Katětov order and uniformity numbers of $M_{\mathscr{I}}$

Lemma

For ideals $\mathscr{I}$, $\mathscr{J}$ on countable sets with $\mathscr{I} \leqslant_K \mathscr{J}$,

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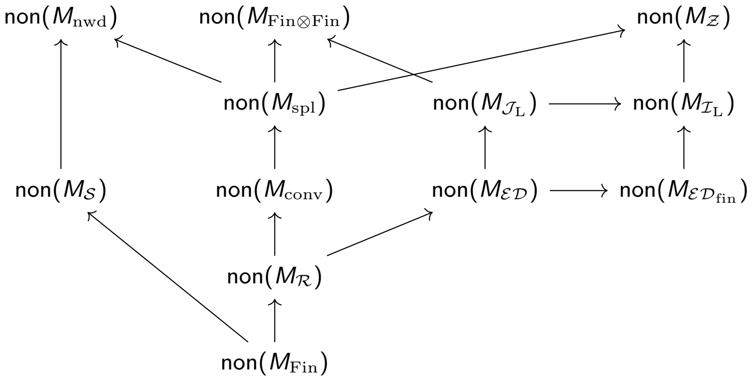

Let $\mathscr{I}$ be an ideal on a countable set X.

Definition

$$\begin{aligned}
\mathrm{add}^*(\mathscr{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I} \wedge \forall B \in \mathscr{I}\, \exists A \in \mathcal{A}\, (A \not\subseteq^* B)\}\\
\mathrm{non}^*(\mathscr{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq [X]^\omega \wedge \forall B \in \mathscr{I}\, \exists A \in \mathcal{A}\, (|A \cap B| < \omega)\}\\
\mathrm{cov}^*(\mathscr{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I} \wedge \forall B \in [X]^\omega\, \exists A \in \mathcal{A}\, (|A \cap B| = \omega)\}\\
\mathrm{cof}^*(\mathscr{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I} \wedge \forall B \in \mathscr{I}\, \exists A \in \mathcal{A}\, (B \subseteq^* A)\}
\end{aligned}$$

$\mathrm{add}^*(\mathscr{I})$ and $\mathrm{non}^*(\mathscr{I})$ badly behave for non-P-ideals $\mathscr{I}$. $\mathrm{add}^*(\mathscr{I}) = \omega$ for non-P-ideals $\mathscr{I}$ and $\mathrm{non}^*(\mathscr{I}) = \omega$ for any Borel ideal $\mathscr{I} \not\geq_{KB} \mathcal{ED}_{\mathrm{fin}}$.

Definition

$$\begin{aligned}
\mathrm{add}^*_\omega(\mathscr{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathscr{I} \wedge \forall \overline{B} \in [\mathscr{I}]^\omega\, \exists A \in \mathcal{A}\, \forall B \in \overline{B}\, (A \not\subseteq^* B)\}\\
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\end{aligned}$$

For P-ideals $\mathscr{I}$, $\mathrm{inv}^*_\omega(\mathscr{I}) = \mathrm{inv}^*(\mathscr{I})$ for $\mathrm{inv} \in \{\mathrm{add}, \mathrm{non}, \mathrm{cof}\}$. These ω-versions of $*$-numbers were introduced by Brendle–Shelah.

ω-versions of $*$-numbers of $\mathscr{I}$

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Relation between cardinal invariants of $\mathscr{I}$ and $M_{\mathscr{I}}$

Theorem

Let $\mathscr{I}$ be an ideal on a countable set. Then the following hold:

1. $\mathrm{add}(M_{\mathscr{I}}) = \mathrm{add}(K_{\mathscr{I}}) = \min\{\mathfrak{b}, \mathrm{add}^*_\omega(\mathscr{I})\}$.
2. $\mathfrak{b} \leqslant \mathrm{non}(K_{\mathscr{I}}) \leqslant \mathrm{non}(M_{\mathscr{I}}) \leqslant \max\{\mathfrak{b}, \mathrm{non}^*_\omega(\mathscr{I})\}$.
3. $\min\{\mathfrak{d}, \mathrm{cov}^*(\mathscr{I})\} \leqslant \mathrm{cov}(M_{\mathscr{I}}) \leqslant \mathrm{cov}(K_{\mathscr{I}}) \leqslant \mathfrak{d}$.
4. $\mathrm{cof}(M_{\mathscr{I}}) = \mathrm{cof}(K_{\mathscr{I}}) = \max\{\mathfrak{d}, \mathrm{cof}^*_\omega(\mathscr{I})\}$.

ideal	$\mathrm{add}^*_\omega(\mathscr{I})$	$\mathrm{non}^*_\omega(\mathscr{I})$	$\mathrm{cov}^*(\mathscr{I})$	$\mathrm{cof}^*_\omega(\mathscr{I})$
$\mathcal{R}$	ω_1	ω_1	$\mathfrak{c}$	$\mathfrak{c}$
$\mathcal{S}$	ω_1	$\mathrm{cov}_\omega(\mathcal{N})$	$\mathrm{non}(\mathcal{N})$	$\mathfrak{c}$
nwd	$\mathrm{add}(\mathcal{M})$	$\mathrm{non}(\mathcal{M})$	$\mathrm{cov}(\mathcal{M})$	$\mathrm{cof}(\mathcal{M})$
conv	ω_1	ω_1	$\mathfrak{c}$	$\mathfrak{c}$
spl	ω_1	$\mathfrak{s}$	$\mathfrak{r}$	$\mathfrak{c}$
$\mathrm{Fin} \otimes \mathrm{Fin}$	$\mathfrak{b}$	$\mathfrak{d}$	$\mathfrak{b}$	$\mathfrak{d}$
$\mathcal{ED}$	ω_1	$\mathrm{cov}(\mathcal{M})$	$\mathrm{non}(\mathcal{M})$	$\mathfrak{c}$
$\mathcal{ED}_{\mathrm{fin}}$	ω_1	?	?	$\mathfrak{c}$
$\mathcal{J}_L$	ω_1	?	?	$\mathfrak{c}$
$\mathcal{I}_L$	ω_1	?	?	$\mathfrak{c}$
$\mathcal{Z}$	$\mathrm{add}(\mathcal{N})$	$\mathrm{non}^*(\mathcal{Z})$	$\mathrm{cov}^*(\mathcal{Z})$	$\mathrm{cof}(\mathcal{N})$

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3. $\min\{\mathfrak{d}, \mathrm{cov}^*(\mathscr{I})\} \leqslant \mathrm{cov}(M_{\mathscr{I}}) \leqslant \mathrm{cov}(K_{\mathscr{I}}) \leqslant \mathfrak{d}$.
4. $\mathrm{cof}(M_{\mathscr{I}}) = \mathrm{cof}(K_{\mathscr{I}}) = \max\{\mathfrak{d}, \mathrm{cof}^*_\omega(\mathscr{I})\}$.

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$\mathcal{ED}$	ω_1	$\mathrm{cov}(\mathcal{M})$	$\mathrm{non}(\mathcal{M})$	$\mathfrak{c}$
$\mathcal{ED}_{\mathrm{fin}}$	ω_1	?	?	$\mathfrak{c}$
$\mathcal{J}_L$	ω_1	?	?	$\mathfrak{c}$
$\mathcal{I}_L$	ω_1	?	?	$\mathfrak{c}$
$\mathcal{Z}$	$\mathrm{add}(\mathcal{N})$	$\mathrm{non}^*(\mathcal{Z})$	$\mathrm{cov}^*(\mathcal{Z})$	$\mathrm{cof}(\mathcal{N})$

(Sabok–Zapletal) The splitting ideal spl is the ideal on $2^{<\omega}$ generated by sets $A \subseteq 2^{<\omega}$ such that there is $c \in [\omega]^\omega$ such that $t \restriction c$ is constant for every $t \in A$.

Theorem (Sabok–Zapletal)

Let $\mathscr{I}$ be a Borel ideal on a countable set.

1. $\mathbb{M}_{\mathscr{I}}$ *adds Cohen reals iff* $\mathrm{nwd} \leqslant_K \mathscr{I} \restriction A$ *for some* $A \notin \mathscr{I}$.
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Corresponding to Sabok–Zapletal's results, we have:

Theorem

1. $\mathrm{non}(M_{\mathrm{nwd}}) = \mathrm{non}(\mathcal{M}), \mathrm{cov}(M_{\mathrm{nwd}}) = \mathrm{cov}(\mathcal{M})$.
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3. $\mathrm{non}(M_{\mathcal{S}}) = \mathrm{non}(K_{\mathcal{S}}) = \max\{\mathfrak{b}, \mathrm{cov}_\omega(\mathcal{N})\}, \mathrm{cov}(M_{\mathcal{S}}) = \mathrm{cov}(K_{\mathcal{S}}) = \min\{\mathfrak{d}, \mathrm{non}(\mathcal{N})\}$.

Here, $\mathrm{cov}_\omega(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{N} \wedge \forall A \in [\mathbb{R}]^\omega \, \exists N \in \mathcal{F}\, (A \subseteq N)\}$. This is consistently different from $\mathrm{cov}(\mathcal{N})$.

Some consequences of Sabok–Zapletal's work

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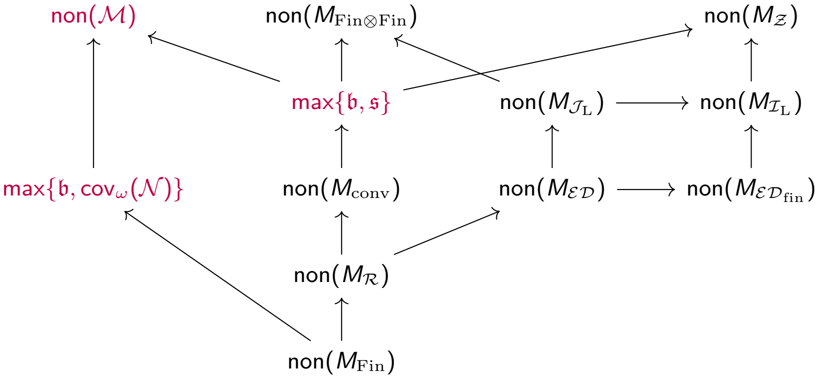
$\mathrm{non}(\mathcal{M})$
$\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$
$\mathrm{non}(M_{\mathcal{Z}})$
$\max\{\mathfrak{b}, \mathfrak{s}\}$
$\mathrm{non}(M_{\mathcal{J}_{\mathrm{L}}})$
$\mathrm{non}(M_{\mathcal{I}_{\mathrm{L}}})$
$\max\{\mathfrak{b}, \mathrm{cov}_{\omega}(\mathcal{N})\}$
$\mathrm{non}(M_{\mathrm{conv}})$
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Other F_σ non-P-ideals

Theorem

1. $\mathrm{non}(M_{\mathrm{conv}}) = \mathrm{non}(M_{\mathcal{R}}) = \mathrm{non}(K_{\mathrm{conv}}) = \mathrm{non}(K_{\mathcal{R}}) = \mathfrak{b}$.
2. $\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{e}^{\mathrm{const}}_{\leqslant}(2)$, $\mathrm{non}(K_{\mathrm{Fin}\otimes\mathrm{Fin}}) = \mathfrak{b}$.
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4. $\mathrm{non}(M_{\mathcal{ED}_{\mathrm{fin}}}) \geqslant \mathfrak{e}^{\mathrm{const}}_b(2)$ *for all increasing* $b \in \omega^\omega$.

Here, $\mathfrak{e}^{\mathrm{const}}(2), \mathfrak{e}^{\mathrm{const}}_b(2), \mathfrak{e}^{\mathrm{const}}_{\leqslant}(2)$ are (variants of) constant evasion numbers, which are defined on the next page.

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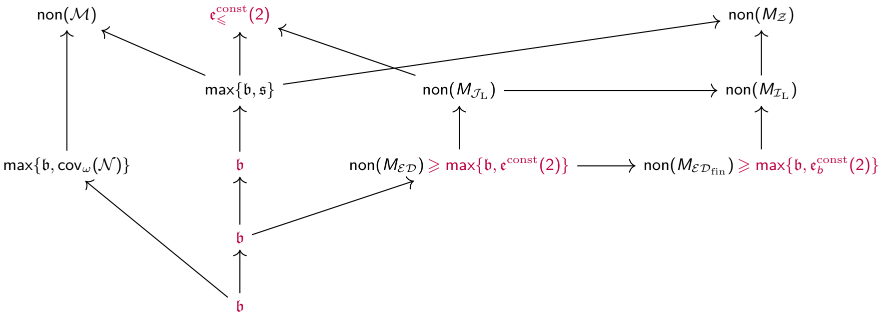

Definition

- *Let* Pred *be the set of all* $\pi\colon \omega^{<\omega} \to \omega$. *For* $b \in (\omega \setminus 2)^\omega$, $\prod b := \prod_{n<\omega} b(n)$ *and*

$$\mathrm{Pred}_b := \left\{\pi \in \mathrm{Pred} : \forall \sigma \in \mathrm{dom}(\pi) \left(\sigma \in \prod_{n<|\sigma|} b(n) \wedge \pi(\sigma) \in b(|\sigma|)\right)\right\}.$$

- *Let* $f \in \omega^\omega$ *and* $\pi \in \mathrm{Pred}$. *For* $k \geqslant 2$, *define*

$$f \sqsubset^k \pi \iff \forall^\infty i < \omega\, \exists j \in [i, i+k)\ f(j) = \pi(f \upharpoonright j)$$
$$f \sqsubset^k_\leqslant \pi \iff \forall^\infty i < \omega\, \exists j \in [i, i+k)\, f(j) \leqslant \pi(f \upharpoonright j)$$

- *Define the following cardinal invariants:*

$$\mathfrak{e}^{\mathrm{const}}(k) := \min\{|F| : F \subseteq \omega^\omega \wedge \forall \pi \in \mathrm{Pred}\, \exists f \in F\, \neg(f \sqsubset^k \pi)\},$$
$$\mathfrak{e}^{\mathrm{const}}_b(k) := \min\{|F| : F \subseteq \prod b \wedge \forall \pi \in \mathrm{Pred}_b\, \exists f \in F\, \neg(f \sqsubset^k \pi)\},$$
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Recall that

$$\mathcal{Z} = \left\{ A \subseteq \omega : \lim_{n\to\infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

Theorem

$$\mathrm{non}(M_{\mathcal{Z}}) = \max\{\mathfrak{b}, \mathrm{non}(\mathcal{E})\}, \mathrm{cov}(M_{\mathcal{Z}}) = \min\{\mathfrak{d}, \mathrm{cov}(\mathcal{E})\}.$$

Asymptotic density zero ideal

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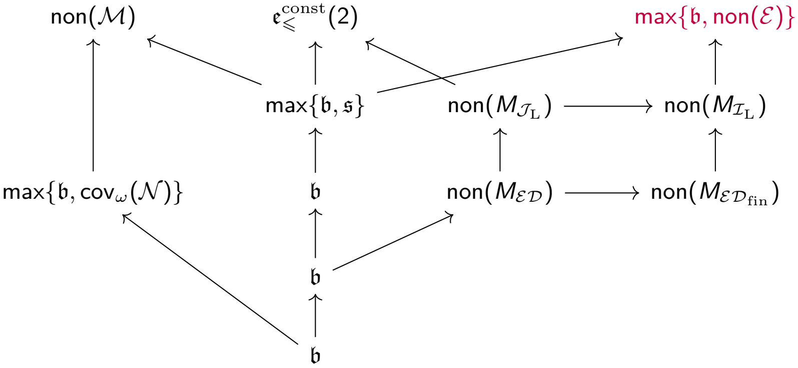

New bounds for $\mathrm{non}^*(\mathcal{Z})$ and $\mathrm{cov}^*(\mathcal{Z})$

Recall that for an ideal $\mathscr{I}$ on a countable set X,

$$\mathrm{non}^*(\mathscr{I}) := \min\{|\mathcal{F}| : \mathcal{F} \subseteq [X]^\omega \land \forall A \in \mathscr{I}\, \exists B \in \mathcal{F}\, |A \cap B| < \omega\},$$
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Corollary

$\max\{\mathfrak{b}, \mathrm{non}(\mathcal{E})\} \leqslant \mathrm{non}^*(\mathcal{Z})$ *and* $\mathrm{cov}^*(\mathcal{Z}) \leqslant \min\{\mathfrak{d}, \mathrm{cov}(\mathcal{E})\}$.

Proof. For any P-ideal $\mathscr{I}$ on ω, $\mathrm{non}(M_{\mathscr{I}}) \leqslant \max\{\mathfrak{b}, \mathrm{non}^*(\mathscr{I})\}$. Also, $\mathfrak{b} \leqslant \mathrm{non}^*(\mathcal{Z})$ (Raghavan–Shelah). Thus,

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$\mathrm{cov}^*(\mathcal{Z}) \leqslant \mathrm{non}^*(\mathcal{Z})$.

Proof. Raghavan showed $\min\{\mathfrak{d}, \mathfrak{r}\} \leqslant \mathrm{non}^*(\mathcal{Z})$ (and $\mathrm{cov}^*(\mathcal{Z}) \leqslant \max\{\mathfrak{b}, \mathfrak{s}(\mathfrak{pr})\}$). Also, it is known that $\mathrm{cov}(\mathcal{E}) \leqslant \mathfrak{r}$. Thus,

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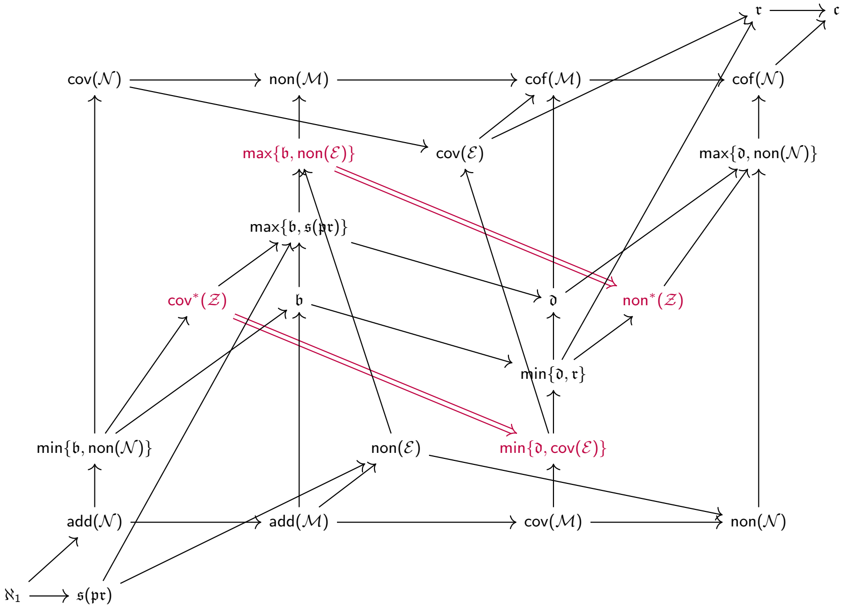
$\mathfrak{r}$
$\mathfrak{c}$
$\mathrm{cov}(\mathcal{N})$
$\mathrm{non}(\mathcal{M})$
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$\mathrm{non}(\mathcal{E})$
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$\aleph_1$
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$\mathcal{ED}, \mathcal{ED}_{\mathrm{fin}}, \mathcal{J}_{\mathrm{L}}, \mathcal{I}_{\mathrm{L}}$

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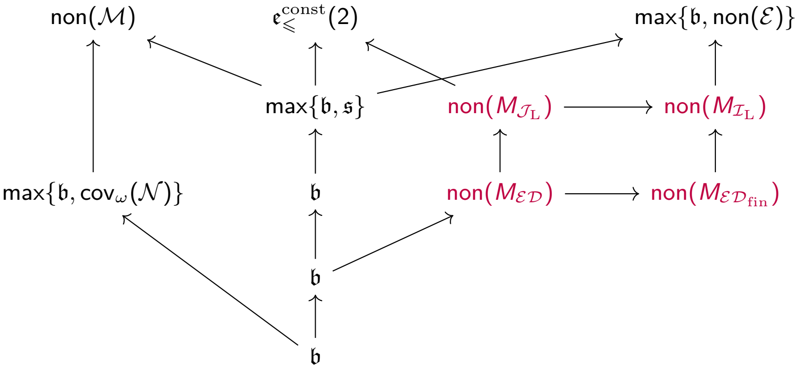

Extended Cichoń's maximum

Recall that $\mathrm{non}(M_{\mathcal{ED}}) \geqslant \max\{\mathfrak{b}, \mathfrak{e}^{\mathrm{const}}(2)\}$ and $\mathrm{non}(M_{\mathcal{ED}_{\mathrm{fin}}}) \geqslant \max\{\mathfrak{b}, \mathfrak{e}_b^{\mathrm{const}}(2)\}$ for all increasing $b \in \omega^\omega$. These inequality cannot be reversed in ZFC:

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Let $\mathscr{I}_0, \mathscr{I}_1$ be ideals on countable sets such that

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Then there is a c.c.c. poset forcing Cichoń's maximum with uniformity and covering numbers of $M_{\mathscr{I}_0}$ and $M_{\mathscr{I}_1}$ (or $K_{\mathscr{I}_1}$).

These consistency results are proved by using (closed) Fr-limits and ultrafilter-limits based on Yamazoe's previous works.

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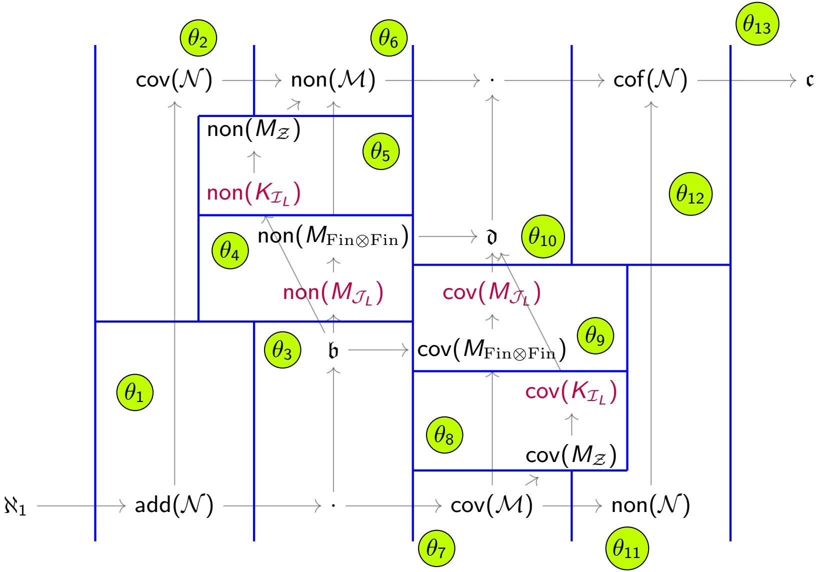
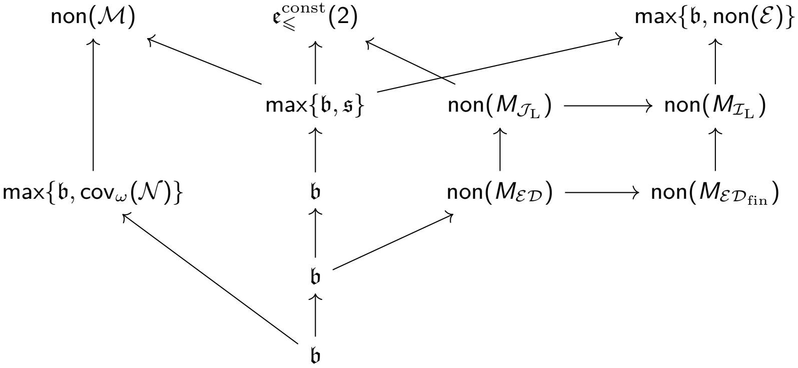


θ_2
θ_6
θ_{13}
$\mathrm{cov}(\mathcal{N})$
$\mathrm{non}(\mathcal{M})$
$\cdot$
$\mathrm{cof}(\mathcal{N})$
$\mathfrak{c}$
$\mathrm{non}(M_{\mathcal{Z}})$
θ_5
$\mathrm{non}(K_{\mathcal{I}_L})$
θ_{12}
$\mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$
$\mathfrak{d}$
θ_{10}
θ_4
$\mathrm{non}(M_{\mathcal{J}_L})$
$\mathrm{cov}(M_{\mathcal{J}_L})$
θ_9
θ_3
$\mathfrak{b}$
$\mathrm{cov}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$
θ_1
$\mathrm{cov}(K_{\mathcal{I}_L})$
θ_8
$\mathrm{cov}(M_{\mathcal{Z}})$
$\aleph_1$
$\mathrm{add}(\mathcal{N})$
$\cdot$
$\mathrm{cov}(\mathcal{M})$
$\mathrm{non}(\mathcal{N})$
θ_7
θ_{11}

Open questions

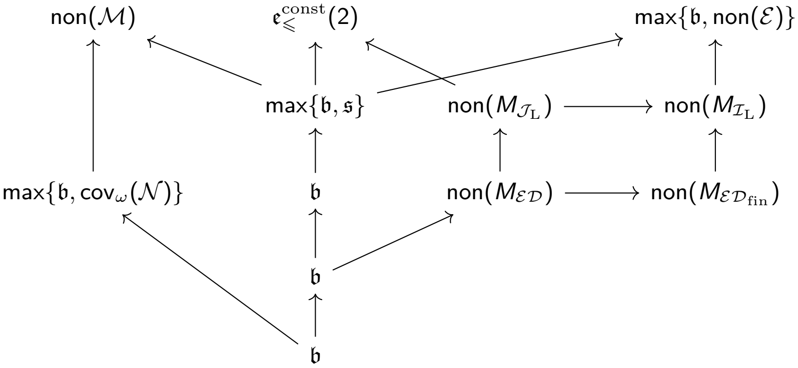
1. Is $\mathrm{non}(K_{\mathscr{I}}) = \mathrm{non}(M_{\mathscr{I}})$ provable in ZFC for $\mathscr{I} = \mathrm{nwd}, \mathcal{ED}_{\mathrm{fin}}, \mathcal{I}_L, \mathcal{Z}$?
2. Are $\mathfrak{b} < \mathrm{non}(M_{\mathcal{ED}}) < \mathrm{non}(\mathcal{M})$ and $\mathfrak{b} < \mathrm{non}(M_{\mathcal{ED}_{\mathrm{fin}}}) < \mathrm{non}(M)$ consistent?
3. Are $\mathrm{non}(M_{\mathcal{ED}}) < \mathrm{non}(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$ and $\mathrm{non}(M_{\mathcal{ED}_{\mathrm{fin}}}) < \mathrm{non}(M_{\mathcal{Z}})$ consistent?



Thank you for your attention!

Open questions

1. Is non$(K_{\mathscr{I}})$ = non$(M_{\mathscr{I}})$ provable in ZFC for $\mathscr{I} = \mathrm{nwd}, \mathcal{ED}_{\mathrm{fin}}, \mathcal{I}_L, \mathcal{Z}$?
2. Are $\mathfrak{b} <$ non$(M_{\mathcal{ED}}) <$ non$(\mathcal{M})$ and $\mathfrak{b} <$ non$(M_{\mathcal{ED}_{\mathrm{fin}}}) <$ non(M) consistent?
3. Are non$(M_{\mathcal{ED}}) <$ non$(M_{\mathrm{Fin}\otimes\mathrm{Fin}})$ and non$(M_{\mathcal{ED}_{\mathrm{fin}}}) <$ non$(M_{\mathcal{Z}})$ consistent?



Thank you for your attention!