A countably tight $P(K)$ space admitting a nonseparable measure

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Today we will focus on countable tightness of $P(K)$.

Definitions

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Let X be a topological space. We say that X has **countable tightness** if for every $A \subseteq X$ and $x \in \overline{A}$ there is a countable $B \subseteq A$ such that $x \in \overline{B}$.

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Question

Is there a countably tight compact space K that carries a nonseparable measure?

The K case

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Haydon '79, Kunen '81, Talagrand '80

Under CH or ♢ there are countably tight compact spaces admitting a nonseparable measure.

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Hope for removing the assumption of MA?

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Construction (Koszmider, S. '24)

Under $\diamondsuit$ there is a compact Hausdorff space K which carries a nonseparable measure but for which $P(K)$ has countable tightness.

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The construction falls into to scheme of the counterexamples given by Haydon, Kunen and Talagrand.

Consequences of the construction and open questions

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1. Can we do the construction using just CH?

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2. Is it consistent that there exists a hereditarily separable (sequential, Frechét-Urysohn) $P(K)$ space with a nonseparable measure?

References:

Koszmider, P. and Silber, Z. (2024), *Countably tight dual ball with a nonseparable measure*. J. London Math. Soc.

Plebanek, G. (2024), *A survey on topological properties of P(K) spaces*. Jpn. J. Math.

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THANKS FOR YOUR ATTENTION.