$\mathcal{I}$-Characterized subgroups of the circle

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Notation and Terminology I [Dikranjan-Impieri, CA, 2014]

Throughout $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ will stand for the set of all real numbers, the set of all rational numbers, the set of all integers and the set of all natural numbers respectively. The first three are equipped with their usual abelian group structure and the circle group $\mathbb{T}$ is identified with the quotient group $\mathbb{R}/\mathbb{Z}$ of $\mathbb{R}$ endowed with its usual compact topology. For $x \in \mathbb{R}$ we denote by $\{x\}$ the difference $x - [x]$ (the fractional part) and $\|x\|$ the distance from the integers i.e. $\min\{\{x\}, 1 - \{x\}\}$.

Topologically torsion elements

The concept of characterized subgroups has evolved significantly over the years, broadening its scope beyond its original foundations to serve as a generalization of the torsion subgroup. Recall that

- An element x of an abelian group G is torsion if there exists $k \in \mathbb{N}$ such that $kx = 0$.

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- Similarly, an element x of an abelian topological group G is :
  - (i) *topologically torsion* if $n!x \to 0$;
  - (ii) *topologically p-torsion*, for a prime p, if $p^n x \to 0$.

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  (ii) *topologically p-torsion*, for a prime p, if $p^n x \to 0$.
- [Armacost, MDN, 1981] defined the subgroups

$$\mathbb{T}_p = \{x \in \mathbb{T} : p^n x \to 0\} \text{ and } \mathbb{T}! = \{x \in \mathbb{T} : n!x \to 0\}.$$

Characterized subgroup [Dikranjan, TP, 2002]

Let (u_n) be a sequence of integers, the subgroup

$$t_{(u_n)}(\mathbb{T}) := \{x \in \mathbb{T} : u_n x \to 0_{\mathbb{T}}\}$$

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The elements of this subgroup are called topologically torsion with respected to the sequence (u_n). And these subgroups were also known as topologically torsion subgroups.
The term *characterized* appeared much later, coined in [Dikranjan, TP, 2002] (as the sequence of integers $\mathbb{Z}$ coincides with the character group of $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, where a character is nothing but a continuous endomorphism of $\mathbb{T}$.)

Motivation

• One motivation for the exploration of the notion of characterized subgroups can be attributed to the examination of sequences of multiples $(a_n\alpha)$ which have deep roots in Number Theory (Weyl's theorem of uniform distribution modulo 1) and in Ergodic Theory (Sturmian sequences and Hartman sets [Winkler, MM, 2002]).

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• These subgroups also play crucial roles in the structure theory of locally compact abelian groups.

• There also happens to be a deep connection with Arbault sets [Arbault, BSMF, 1952] which appear in Harmonic Analysis.

An important class of sequences

In this talk we are interested in characterized subgroups coming from arithmetic sequences.

- **Arithmetic sequence**: A sequence of positive integers (a_n) is called an arithmetic sequence if

$$1 \leq a_1 < a_2 < a_3 < \ldots < a_n < \ldots \quad \text{and } a_n | a_{n+1} \text{ for every } n \in \mathbb{N},$$

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- **Example**: (2^n) with ratio sequence (2), (p^n) with ratio sequence (p), $(n!)$ with ratio sequence (n).

- Note that, $A \subseteq \mathbb{N}$ is called
  - (i) b-bounded if the sequence of ratios $(b_n)_{n \in A}$ is bounded.
  - (ii) b-divergent if the sequence of ratios $(b_n)_{n \in A}$ diverges to ∞.

Notation and Terminology [Dikranjan-Impieri, CA, 2014]

- **Fact**: For any arithmetic sequence (a_n) and $x \in \mathbb{T}$, we can build a unique sequence of integers (c_n), where $0 \leq c_n < b_n$ ($b_n = \frac{a_n}{a_{n-1}}$), such that

$$x = \sum_{n=1}^{\infty} \frac{c_n}{a_n} \qquad \text{and } c_n < b_n - 1 \text{ for infinitely many } n. \qquad (1)$$

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- For $x \in \mathbb{T}$ with canonical representation (1), we define
  - $supp(x) = \{n \in \mathbb{N} \ : \ c_n \neq 0\}$;
  - $supp_b(x) = \{n \in \mathbb{N} \ : \ c_n = b_n - 1\}$.

Observations related to characterized subgroup

In 1952, Eggleston was working with sets having fractional dimension and surprisingly there he have shown that:

[Eggleston, PLMS, 1952]

For an increasing sequence of integers (u_n),

(E1) $t_{(u_n)}(\mathbb{T})$ is countable if $(\frac{u_n}{u_{n-1}})$ is bounded,

(E2) $|t_{(u_n)}(\mathbb{T})| = \mathfrak{c}$ if $(\frac{u_n}{u_{n-1}})$ diverges to ∞.

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Finally, in 2014, Dikranjan and Impieri have shown that a dichotomic version of Eggleston's result holds true for arithmetic sequences!

Dichotomy [Dikranjan-Impieri, CA, 2014]

For an arithmetic sequence of integers (a_n),

(D1) $t_{(a_n)}(\mathbb{T})$ is countable if (a_n) is b-bounded,

(D2) $|t_{(a_n)}(\mathbb{T})| = \mathfrak{c}$ if (a_n) is not b-bounded.

Generalized convergence

- **Ideal**: A non-empty family $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is called an ideal on $\mathbb{N}$ whenever
  - $\mathbb{N} \notin \mathcal{I}$,
  - if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,
  - $A \subset B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$.

  $\mathcal{I}^*$ denotes the dual filter. An ideal $\mathcal{I}$ is called a *P*-ideal if for every sequence (A_n) of sets in $\mathcal{I}$ there is a set $A \in \mathcal{I}$ such that $A_n \subset^* A$ for all $n \in \mathbb{N}$.

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- **Ideal convergence** [Kostyrko, Salat, wilczynsky, RAE, 2000-2001]: In a topological space X, given an ideal $\mathcal{I}$, we say that a sequence (x_n) is $\mathcal{I}$-convergent to $x \in X$ whenever for every open set U containing x, the set $\{n \in \mathbb{N} : x_n \notin U\} \in \mathcal{I}$ (we will write $x_n \to x$ w.r.t $\mathcal{I}$).

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- It is known that "(x_n) is $\mathcal{I}$-convergent to $x \in X$ iff there exists $A \in \mathcal{I}^*$ such that $(x_n)_{n \in A}$ usually converges to x" $\iff$ "$\mathcal{I}$ is a P-ideal"

An overview of generalized characterized subgroups

- **$\mathcal{I}$-characterized subgroup** [Das-Ghosh, APAL, 2023]: Let (a_n) be a sequence of integers, the subgroup

$$t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ w.r.t. } \mathcal{I} \text{ in } \mathbb{T}\}$$

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• Fin		$\Rightarrow$	$t_{(a_n)}(\mathbb{T})$
• $\mathcal{I}_d$		$\Rightarrow$	$t^s_{(a_n)}(\mathbb{T})$
• $\mathcal{I}_g$		$\Rightarrow$	$t^g_{(a_n)}(\mathbb{T})$
• $\mathcal{I}_1$	(Translation invariant)	$\Rightarrow$	$t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T})$
• $\mathcal{I}_2$	$(non - snt)$	$\Rightarrow$	$t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$

simple density ideal

simple density function

$$\underline{d}_g(A) = \liminf_{n\to\infty} \frac{|A \cap [1,n]|}{g(n)} \text{ and } \overline{d}_g(A) = \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{g(n)}. \quad (2)$$

Here g is a non-decreasing weight functions $g : \mathbb{N} \to [0, \infty)$ such that $\lim_{n\to\infty} g(n) = \infty$ and $\frac{n}{g(n)} \nrightarrow 0$.

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$$\mathcal{I}_g = \{A \subset \mathbb{N} : d_g(A) = 0\}.$$

For $g(n) = n$ we have the usual notion of natural density.

Generalized characterized subgroups I

s-characterized Subgroup [Dikranjan-Das-Bose, FM, 2019]

Let (a_n) be a sequence of integers, the subgroup

$$t^s_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \text{ statistically in } \mathbb{T}\}$$

of $\mathbb{T}$ is called an s-characterized (by (a_n)) subgroup of $\mathbb{T}$.

Generalized characterized subgroups II

g-characterized subgroups (special case of [Das-Ghosh, IM, 2021])

For a sequence of integers (a_n) the subgroup

$$t^{g}_{(a_n)}(\mathbb{T}) := \{x \in \mathbb{T} : a_n x \to 0 \ g\text{-statistically in } \mathbb{T}\}$$

of $\mathbb{T}$ is called *an g-statistically characterized* (shortly, *an g-characterized*) (by (a_n)) *subgroup* of $\mathbb{T}$.

Cardinality, Borel Complexity and Novelty

- For any sequence of integers (a_n), $t^s_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.

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- For any arithmetic sequence (a_n), $t^s_{(a_n)}(\mathbb{T}) \supsetneq t_{(a_n)}(\mathbb{T})$.
- For any arithmetic sequence (a_n), the subgroup $t^s_{(a_n)}(\mathbb{T})$ cannot be characterized by any sequence of integers.

Problem Formulation

Problem 1

Let (a_n) be an arithmetic sequence such that $(\frac{a_n}{a_{n-1}})$ is bounded. Does there exist an ideal $\mathcal{I} \neq Fin$ such that $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is countable?

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Problem 2

Let (a_n) be an arithmetic sequence. Does $\mathcal{I}_1 \subsetneq \mathcal{I}_2$ imply $t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$?

We will show that this is not true in general but for a special class of ideals namely translation invariant P-ideals (defined later), the answer to this problem is positive.

Notation and Terminology I

For two subsets A, B of $\mathbb{N}$ and an ideal $\mathcal{I}$, we will write

- $A \subseteq^{\mathcal{I}} B$ if $A \setminus B \in \mathcal{I}$,
- $A \subset \mathbb{N}$ is called $\mathcal{I}$-translation invariant if $A + n = \{m + n \in \mathbb{N} : \ m \in A\}$ belongs to $\mathcal{I}$ for all $n \in \mathbb{Z}$.

Note that $\mathcal{I}$ is called translation invariant if every $A \in \mathcal{I}$ is $\mathcal{I}$-translation invariant.

Notation and Terminology II

Further every ideal $\mathcal{I}$ on $\mathbb{N}$ can be treated as a subset of the Cantor space $2^{\mathbb{N}}$ in view of the fact that $\mathcal{P}(\mathbb{N})$ and $2^{\mathbb{N}}$ can be identified via the characteristic functions. An ideal $\mathcal{I}$ is called analytic if it corresponds to an analytic subset of the Cantor space $2^{\mathbb{N}}$.

snt-ideal

An ideal $\mathcal{I}$ will be called strongly non-translation invariant (in short *snt*-ideal) if it does not contain any infinite $\mathcal{I}$-translation invariant set, i.e., for each infinite subset $A \in \mathcal{I}$, there exists at least one $k \in \mathbb{Z}$ for which $A - k \notin \mathcal{I}$.

Cardinality and Borel Complexity [Ghosh, RM, 2022]

Consider any translation invariant analytic P-ideal $\mathcal{I}$ ($\neq$ *Fin*) of $\mathbb{N}$.

- For any sequence of integers (a_n), $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is an $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.
- For any arithmetic sequence (a_n), we have $|t^{\mathcal{I}}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.
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- For any arithmetic sequence (a_n), we have $|t^{\mathcal{I}}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.
- For any arithmetic sequence (a_n), $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \supsetneq t_{(a_n)}(\mathbb{T})$.
- For any arithmetic sequence (a_n), if $\mathcal{I}_1 \subsetneq \mathcal{I}_2$ then $t^{\mathcal{I}_1}_{(a_n)}(\mathbb{T}) \subsetneq t^{\mathcal{I}_2}_{(a_n)}(\mathbb{T})$.

For a non-negative admissible matrix $\mathbf{M} = (a_{i,j})$ (where admissibility means $\lim_{i\to\infty} a_{i,j} = 0$ for every $j \in \mathbb{N}$) one can define the corresponding upper **M**-density of $A \subset \mathbb{N}$ by

$$\overline{d}_{\mathbf{M}}(A) = \limsup_{n\to\infty} \left(\mathbf{M} \cdot \chi(A)\right)$$

where $\chi(A)$ is the characteristic sequence of A, i.e.,

$$\chi(A) = (x_n) \quad \text{with} \quad x_n = \begin{cases} 1 & \text{if } n \in A, \\ 0 & \text{otherwise.} \end{cases}.$$

Method used here for constructing analytic P-ideals II

If for $A \subset \mathbb{N}$, $\overline{d}_{\mathbf{M}}(A) = \underline{d}_{\mathbf{M}}(A)$ then we say that the **M**-density of A exists and is denoted by $d_{\mathbf{M}}(A)$.

The analytic P-ideal $\mathcal{I}$ corresponding to **M** is obtained by

$$\mathcal{I} = \{A \subset \mathbb{N} : d_{\mathbf{M}}(A) = 0\}.$$

Some examples of *snt*-ideals

Consider the non-negative regular matrix **M** defined by

$$\mathbf{M} = (b_{pq}) \text{ and } b_{pq} = \begin{cases} 1 & \text{if } q = 2p - 1 \\ 0 & \text{otherwise,} \end{cases}$$

The corresponding ideal $\mathcal{I} = \{A \subset \mathbb{N} : d_{\mathbf{M}}(A) = 0\}$ satisfies the property that

$$\mathcal{I} = \{A \subset \mathbb{N} : A \subseteq^* 2\mathbb{N}\}.$$

Now consider any infinite set $B \in \mathcal{I}$. Observe that $B \subseteq^* 2\mathbb{N}$ which implies $B + 1 \notin \mathcal{I}$. Hence the ideal $\mathcal{I}$ is a *snt*-ideal.

Similarly, for any $m \in \mathbb{N} \setminus \{1\}$ if we define

$$\mathcal{I} = \{A \subset \mathbb{N} : A \subseteq^* \mathbb{N} \setminus m\mathbb{N}\}$$

then $\mathcal{I}$ becomes a *snt* analytic *P*-ideal.

Some examples of non-*snt* ideals

Consider the analytic P-ideal $\mathcal{I} = \{A \subset \mathbb{N} : d_{\mathbf{M}}(A) = 0\}$, where **M** is a non-negative regular matrix defined by

$$\mathbf{M} = (b_{ij}) \quad \text{and } b_{ij} = \begin{cases} 1 & \text{if } j = (2i)^2 \\ 0 & \text{otherwise.} \end{cases}$$

Then, the set $A = \{n \in \mathbb{N} \ : n = (2k+1)^2$ for all $k \in \mathbb{N}\}$ is translation invariant w.r.t $\mathcal{I}$. Therefore $\mathcal{I}$ is a non-*snt* ideal. Note that all the density ideals we have studied so far are translation invariant therefore non-snt.

Characterization of *snt*-ideals

The concept of *snt*-ideal comes from the fact that we want ideals that does not have sets with arbitrary block length. Equivalently, we want to construct a filter which do not contain sets with arbitrary gap in it.

An analytic P-ideal $\mathcal{I}$ is a *snt*-ideal if and only if
$(**)$ for all $A^* = \{n_1 < n_2 < \ldots < n_i < \ldots\} \in \mathcal{I}^*$ there exists a fixed $m \in \mathbb{N}$ and $i_{A^*} \in \mathbb{N}$ such that $n_{i+1} - n_i \leq m$ for all $i \geq i_{A^*}$.

Main Results [Das-Ghosh, APAL, 2023] I

Theorem 1

For any analytic P-ideal $\mathcal{I}$, $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is a $F_{\sigma\delta}$ (hence, Borel) subgroup of $\mathbb{T}$ containing $t_{(a_n)}(\mathbb{T})$.

Theorem 2

For any analytic P-ideal $\mathcal{I}$ and $x \in \mathbb{T}$, if $supp(x) \in \mathcal{I}$ and $supp(x)$ is $\mathcal{I}$-translation invariant, then $x \in t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$.

Main Results [Das-Ghosh, APAL, 2023] II

Theorem 3

For any non-*snt* analytic P-ideal $\mathcal{I}$ and for any arithmetic sequence (a_n), $|t^{\mathcal{I}}_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.

Theorem 4

For any non-*snt* analytic P-ideal $\mathcal{I}$ and for any arithmetic sequence (a_n), $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \supsetneq t_{(a_n)}(\mathbb{T})$ and $|t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) \setminus t_{(a_n)}(\mathbb{T})| = \mathfrak{c}$.

Therefore, all the cardinality related observations (that we have seen for translation invariant ideals) actually hold for non-snt ideals.

Main Results [Das-Ghosh, APAL, 2023] III

Theorem 5

For a *b*-bounded arithmetic sequence (a_n) and an analytic *P*-ideal $\mathcal{I}$, $t^{\mathcal{I}}_{(a_n)}(\mathbb{T}) = t_{(a_n)}(\mathbb{T})$ if and only if $\mathcal{I}$ is a *snt*-ideal.

This result provides a negative solution to Problem 2 (as $Fin \subsetneq \mathcal{I}$).

Theorem 6

Let $\mathcal{I}$ be a *snt* analytic *P*-ideal. Then $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ is countable if and only if (a_n) is *b*-bounded.

This provides a positive solution to problem 1, i.e., there exists an ideal strictly bigger than the ideal *Fin* for which the subgroup $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ can be countable.

Main Results [Das-Ghosh, APAL, 2023] IV

Novelty [Das-Ghosh, Communicated, 2025]

For any dense analytic P-ideal $\mathcal{I}$ and for any arithmetic sequence (a_n), the subgroup $t^{\mathcal{I}}_{(a_n)}(\mathbb{T})$ cannot be characterized by any sequence of integers.

Conclusion and Future Scope

Open Problem 1

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Open Problem 3

Compare the class of non-snt ideals with other well known class of ideals (in particular, the class of dense ideals).

Reference I

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