Universally meager sets in the Miller model and similar ones

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Convention:

All spaces are metrizable, separable and zero-dimensional, i.e., subspaces of 2^ω up to homeomorphisms.

A space is **totally imperfect**, if it contains no copy of 2^ω.

Motivation:

Theorem (H., Szewczak, Zdomskyy, 2023)

In the Sacks model, every Menger space $X \subseteq 2^\omega$ has size at most $\mathfrak{d} = \omega_1 < \mathfrak{c} = \omega_2$ or contains a copy of 2^ω.

Theorem (Bartoszyński, Tsaban, 2006)

There exists a totally imperfect Menger space of size $\mathfrak{d}$.

Theorem (H., Szewczak, Zdomskyy, 2024)

In the Miller model, there are no γ-sets $X \subseteq 2^\omega$ of size $\mathfrak{c}$ and there are no concentrated sets $X \subseteq 2^\omega$ of size $\mathfrak{c}$.

Note: $\mathfrak{d} = \mathfrak{c}$ in the Miller model, so Menger not suitable for "perfect set property". Is Hurewicz suitable?

Definition

$X \subseteq 2^\omega$ is **concentrated** on $A \subseteq 2^\omega$ with $|A| = \omega$ if for any open $U \supseteq A$: $|X \setminus U| \leq \omega$. Moreover, we call X *concentrated* if $A \subseteq X$.

γ-sets ($\mathsf{S}_1(\Omega, \Gamma)$):
For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of ω-covers there are sets $U_n \in \mathcal{U}_n$, $n \in \omega$, such that $\{ U_n : n \in \omega \}$ is a γ-cover of X, i.e. for all $x \in X$: $|\{ n \in \omega : x \notin U_n \}| < \omega$.

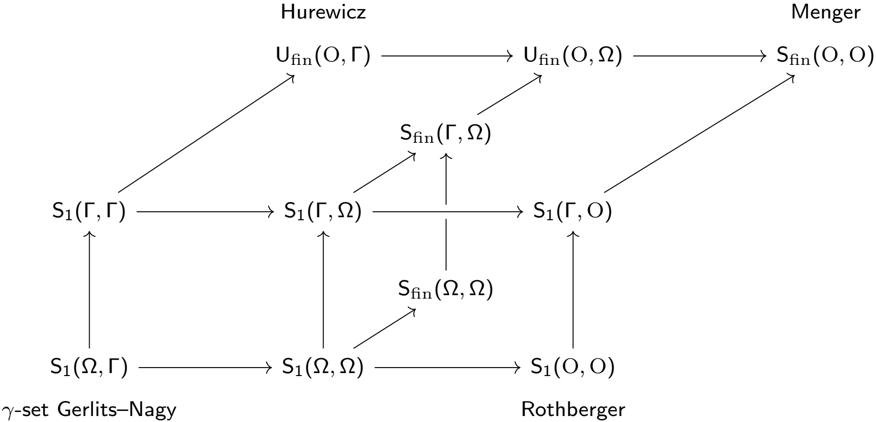
Remark

Both are examples of Rothberger spaces.

Selection principles

Menger spaces ($\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$)**:** For every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is an open cover of X.
Hurewicz spaces ($\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$)**:** For every sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers of X there is a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ and $\{\cup \mathcal{V}_n : n \in \omega\}$ is a γ-cover of X, i.e. for all $x \in X$:
$|\{ n \in \omega : x \notin \bigcup \mathcal{V}_n \}| < \omega$.
Rothberger spaces ($\mathsf{S}_1(\mathrm{O},\mathrm{O})$)**:**
For each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there is a sequence $\langle U_n : n \in \omega \rangle$ such that $U_n \in \mathcal{U}_n$, and $\{ U_n : n \in \omega \}$ is an open cover of X.

Scheepers diagram



The Scheepers Diagram illustrating the connections among the selection principles, excluding trivial ones or those that are equivalent in ZFC.

Miller model

$T \subseteq \omega^{<\omega}$ is a **Miller tree** if T is closed under initial segments; and for every $t \in T$ there is $s \supseteq t$ such that $s^\frown n \in T$ for infinitely many $n \in \omega$.

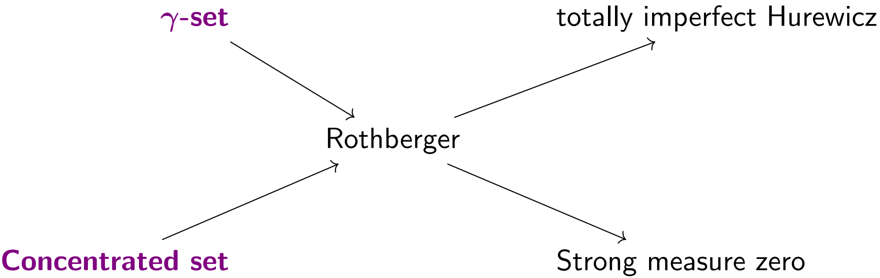
- Miller forcing: $\mathbb{M} = \{T \subseteq \omega^{<\omega} : T \text{ is a Miller tree}\}$ with $\leq := \subseteq$.
- $\mathbb{M}_{\omega_2}$: the countable support iteration (c.s.i.) of Miller forcing of length ω_2.

Miller model: Forcing with $\mathbb{M}_{\omega_2}$ over a model of CH. Some properties:

- $\mathfrak{c} = \mathfrak{d} = \omega_2$
- $\mathfrak{b} = \omega_1$
- $\mathfrak{u} < \mathfrak{g}$
- There are totally imperfect Menger subspaces of reals of size $\mathfrak{c}$.

Theorem (Zdomskyy, 2005)

If $\mathfrak{u} < \mathfrak{g}$, then every Rothberger space is totally imperfect Hurewicz.



Question

In the Miller model, does every Hurewicz space $X \subseteq 2^\omega$ have size at most $\mathfrak{b} = \omega_1 < \mathfrak{c} = \omega_2$ or contain a copy of 2^ω?

Universally meager sets

First studied by Grzegorek in 1984 as the notion of **absolutely of the first category sets:** cannot contain Borel isomorphic copies of non-meager sets.

Definition (Zakrzewski)

$X \subseteq 2^\omega$ is **universally meager** if for every Borel isomorphism $f : 2^\omega \to 2^\omega$ the image $f[X]$ is meager in 2^ω.

Theorem (Zakrzewski, 2008)

$X \subseteq 2^\omega$ is universally meager if and only if for every Polish space Y and continuous nowhere constant map $f : Y \to 2^\omega$ the preimage $f^{-1}[X]$ is meager in Y.

Proposition (Zakrzewski, 2001)

If $X \subseteq 2^\omega$ is totally imperfect Hurewicz, then X is universally meager.

Property (†)

Definition

Let $\mathbb{P}$ be a forcing notion. $\mathbb{P}$ **satisfies (†):** countable elementary submodel $M \ni \mathbb{P}, \ldots$ of $H(\theta)$ for θ large enough. For every sequence $\langle \varphi_n : n \in \omega \rangle$ of functions from $M \cap \mathbb{P}$ to $M \cap \mathbb{P}$ with $\forall r\ (\varphi_n(r) \leq r)$: $\forall p_0 \in M \cap \mathbb{P}$ $\exists q \leq p_0$ $(M, \mathbb{P})$-generic condition s.t.

$$\forall n \in \omega : \quad q \Vdash \text{"}\Gamma_{\mathbb{P}} \cap \varphi_n[M \cap \mathbb{P}] \neq \emptyset\text{"}.$$

Proposition

$\mathbb{P}$ satisfies (†) iff for every $\varphi \colon M \cap \mathbb{P} \to M \cap \mathbb{P}$ with $\forall r\ (\varphi(r) \leq r)$: $\forall p_0 \in M \cap \mathbb{P}$ $\exists q \leq p_0$ $(M, \mathbb{P})$-generic condition s.t.

$$q \Vdash \text{"}|\Gamma_{\mathbb{P}} \cap \varphi[M \cap \mathbb{P}]| = \omega\text{"}.$$

Property (†)

Proposition

A forcing notion with property (†) cannot add eventually different reals.
(i.e. $g \in \omega^\omega$ s.t. $\forall f \in V \cap \omega^\omega$: $|\{ n \in \omega : f(n) = g(n) \}| < \omega$)

Proof.

- τ name for an element of ω^ω and $p \in \mathbb{P}$
- Take $M \ni \tau, p$ and write $M \cap \mathbb{P} = \{ p_j : j \in \omega \}$.
- Pick $\varphi(p_j) \leq p_j$ and $a_j \in \omega$ s.t. $\varphi(p_j) \Vdash$ "$\tau(j) = a_j$" and set $g_M \in \omega^\omega \cap V$ to be $g_M(j) = a_j$.
- $(\dagger) \Rightarrow \exists q \leq p$ s.t. $q \Vdash$ "$\exists^\infty r \in M \cap \mathbb{P}\ (\varphi(r) \in \Gamma_{\mathbb{P}})$".
- Pick generic filter $G \subseteq \mathbb{P}$ and by density, find such $M, q \in G$.
- $|\{ j \in \omega : \tau^G(j) = a_j = g_M(j) \}| = \omega$.

□

Property (†)

(†) cannot add eventually different reals, so in particular, it cannot add dominating reals.

Example

Random model, Hechler model, Laver model do not satisfy (†).

However:

Example

Cohen forcing, Sacks forcing, Miller forcing (single step) all satisfy (†).

Lemma

Let $\mathbb{P}$ be a forcing with property (†) and let $\dot{\mathbb{Q}}$ be a name for a forcing with property (†). Then $\mathbb{P} * \dot{\mathbb{Q}}$ satisfies (†).

Funfact: Generalize preservation result for (†)

Theorem (Miller, Tsaban, Zdomskyy, 2016)

After adding one Cohen real, a ground model Hurewicz space $X \subseteq 2^\omega \cap V$ is Hurewicz in the extension iff it is Rothberger in the ground model.

Proposition

After forcing with $\mathbb{P}$ satisfying (†), if a ground model Hurewicz space $X \subseteq 2^\omega \cap V$ is Rothberger, than it is Hurewicz and Rothberger in the extension.

Main result

Theorem (H., Szewczak, Zdomskyy, 2025)

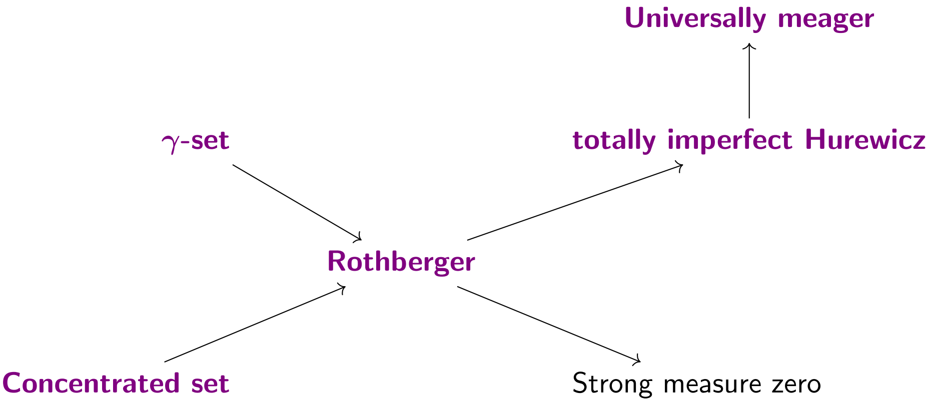
Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \delta \rangle$ *be a countable support iteration of forcings satisfying property* $(\dagger)$ *with limit* $\mathbb{P}_\delta$*. Then* $\mathbb{P}_\delta$ *satisfies* $(\dagger)$*.*

Theorem (H., Szewczak, Zdomskyy, 2025)

If $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ *is a c.s.i. with limit* $\mathbb{P}$ *over a model of CH such that for all* $\alpha < \omega_2$

$$\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \text{"} |\dot{\mathbb{Q}}_\alpha| \leq \omega_1 \textit{ and } \dot{\mathbb{Q}}_\alpha \textit{ satisfies } (\dagger) \text{``},$$

then in the generic extension all universally meager subsets of 2^ω *are of size at most* ω_1*.*



Corollary

In the Miller model, all totally imperfect Hurewicz spaces and all Rothberger spaces have size at most $\omega_1 < \mathfrak{c}$.

All universally meager sets are **perfectly meager**. (i.e. for all perfect sets $P \subseteq 2^\omega$, $X \cap P$ is meager in P)

Theorem (Bartoszyński, 2002)

In the Miller model, a set is universally meager iff it is perfectly meager.

Corollary

In the Miller model, all perfectly meager sets $X \subseteq 2^\omega$ have size at most ω_1.

Application for the Borel Conjecture

Recall, a space is **Rothberger** if for each sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of open covers there is a sequence $\langle U_n : n \in \omega \rangle$ such that $U_n \in \mathcal{U}_n$, and $\{ U_n : n \in \omega \}$ is an open cover of X.

Theorem (Miller, 2005)

There exists a strong measure zero set $X \subseteq 2^\omega$ of size ω_1 iff there exists a Rothberger space $Y \subseteq 2^\omega$ of size ω_1. In particular, the Borel conjecture is equivalent to the "Borel conjecture for the Rothberger property".

Question

Does the above also hold for ω_2?

The Goldstern-Judah-Shelah model

The **Goldstern-Judah-Shelah model** is a c.s.i. $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha < \omega_2 \rangle$ of length ω_2 over a ground model of CH. In particular, it has the form:

1. $\forall \alpha < \omega_2 \ \left(\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{M}}\text{”}\right.$ or $\exists \dot{H} \ \left(\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\dot{\mathbb{Q}}_\alpha = \mathrm{PT}_{\dot{H}}\right)\big)\text{”}$,
2. For cofinally many $\alpha < \omega_2$, $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\dot{\mathbb{Q}}_\alpha = \dot{\mathbb{M}}\text{”}$,
3. For cofinally many $\alpha < \omega_2$, $\exists \dot{H} \ \left(\mathbb{1}_{\mathbb{P}_\alpha} \Vdash \text{“}\dot{\mathbb{Q}}_\alpha = \mathrm{PT}_{\dot{H}}\right)\big)\text{”}$,

where $\dot{\mathbb{M}}$ is a name for the Miller forcing in the corresponding generic extension.

PT_H

Definition

$H : \omega \to \mathcal{P}(\omega)$ a function s.t. for all $n \in \omega$ holds $1 < |H(n)| < \omega$.
The **forcing notion** PT_H is ordered by inclusion and consists of trees $p \subseteq \omega^{<\omega}$ such that

1. $\forall \eta \in p\, \forall l \in \mathrm{dom}(\eta)\ \ (\eta(l) \in H(l))$,
2. $\forall \eta \in p\ \ (|\mathrm{succ}_p(\eta)| \in \{1, |H(|\eta|)|\})$,
3. $\forall \eta \in p\, \exists \nu \in p\ \ (\eta \subseteq \nu \text{ and } \mathrm{succ}_p(\nu) = H(|\nu|))$.

Proposition

PT_H is a proper ω^ω-bounding forcing that satisfies $(\dagger)$ and $|PT_H| \leq \mathfrak{c}$.

The Goldstern-Judah-Shelah model

Theorem (Goldstern, Judah, Shelah, 1993)

In the G-J-S model, there exists a strong measure zero set $X \subseteq 2^\omega$ of size $\omega_2 = \mathfrak{c}$.

If we want to apply our results, we have to make sure that Rothberger spaces are universally meager in this model. Goldstern, Judah and Shelah showed that in their model P-points are preserved. Hence, $\mathfrak{u} = \omega_1$.

Proposition

In the G-J-S model, $\mathfrak{u} < \mathfrak{g}$. In particular, Rothberger spaces are universally meager.

Consistency results witnessed by the G-J-S model

Theorem (H., Szewczak, Zdomskyy, 2025)

It is consistent with ZFC that there exists a strong measure zero set $X \subseteq 2^\omega$ of size ω_2 and all Rothberger spaces $Y \subseteq 2^\omega$ have size at most ω_1.

Proposition

If every universally meager space is SMZ, then every Hurewicz totally imperfect space is Rothberger

Fact

In the G-J-S model, $[2^\omega]^{<\mathfrak{c}} \subseteq$ SMZ.

Theorem (H., Szewczak, Zdomskyy, 2025)

It is consistent with ZFC that totally imperfect Hurewicz is equivalent with Rothberger.

Miller model: weak Borel conjecture

In the construction of the G-J-S model $PT_{\dot{H}}$ was needed to get a strong measure zero set of size $\mathfrak{c}$. Did the Miller forcing not suffice?

Theorem (Galvin, Mycielski, Solovay, 1973)

$X \subseteq 2^\omega$ is strong measure zero iff for every meager subset $F \subseteq 2^\omega$ holds that $X + F \neq 2^\omega$.

Definition (Prikry)

$X \subseteq 2^\omega$ is called strongly meager if for every strong measure zero set $H \subseteq 2^\omega$ holds $X + H \neq 2^\omega$.

weak Borel conjecture: $\mathrm{SMZ} \subseteq [2^\omega]^{<\mathfrak{c}}$.

weak dual Borel conjecture: strongly meager $\subseteq [2^\omega]^{<\mathfrak{c}}$.

Miller model: weak Borel conjecture

Theorem (Bartoszyński, Shelah, 2003)

The Miller model and the Laver model fulfill the weak dual Borel Conjecture.

The motivation of their paper was whether the Borel Conjecture and Dual Borel Conjecture are jointly consistent (proven by Goldstern, Kellner, Shelah and Wohofsky in 2014) and in particular, how strongly meager sets look in various models.

Question

Does the Laver model fulfill the Dual Borel Conjecture?

Question (Wohofsky)

Does the Miller model fulfill the weak Borel conjecture?

Miller model: weak Borel conjecture

Theorem (H., Szewczak, Zdomskyy, 2025)

In the Miller model, the weak Borel conjecture holds.

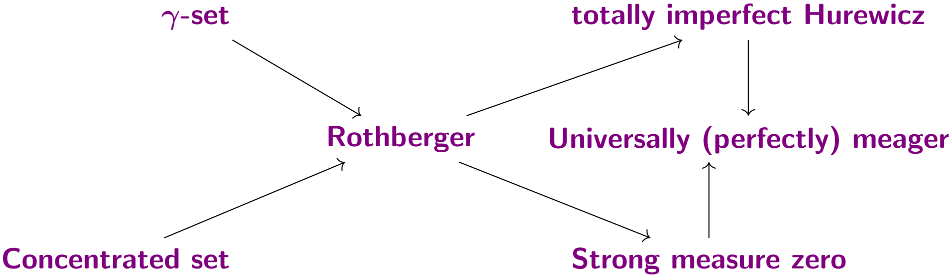
Rough idea.

- c.s.i. of length ω_2 over CH, where ω_1-stationary often the Miller forcing is iterated and every iterand fulfills the Laver property, i.e. for $f \in \omega^\omega \cap V[G]$ with $f \leq h \in V$, there is $\langle H(n) : n \in \omega \rangle \in V$ s.t. $\forall n \in \omega : H(n) \in [h(n)]^n$ and $f(n) \in H(n)$.
- Then every strong measure zero set $X \subseteq 2^\omega$ in the extension is meager.
- Every strong measure zero meager set must be perfectly meager.

□

Summary:

In the Miller model, all properties below are only witnessed by spaces of size $< \mathfrak{c}$.



Open Problems:

Problem

Is there a totally imperfect Hurewicz space which is not Rothberger, in the Miller model?

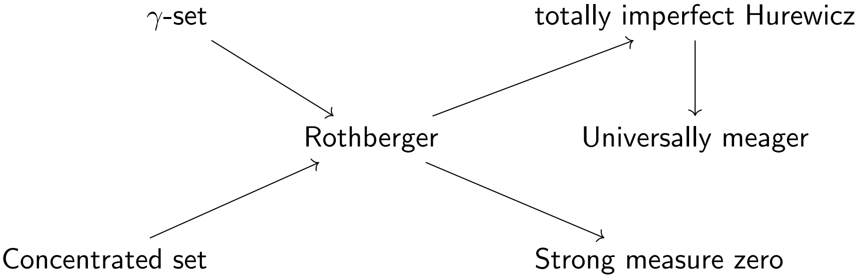
Problem

Does a finite support iteration of ccc forcings satisfying property $(\dagger)$ *satisfy property* $(\dagger)$*?*

Open Problems:

Problem

What is possible in models of $\mathfrak{u} < \mathfrak{g}$ where the forcing does not satify $(\dagger)$?



Thank you for your attention!