Laver ultrafilters

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Joint work with Tan Özalp

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Introduction

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Note:

- If $\mathbb{P}$ has the Laver property, then $\mathbb{P}$ does not add Cohen reals.
- The Laver property is preserved under countable support iterations of proper forcing notions.

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Let $\mathcal{U}$ be an ultrafilter over ω. The forcing notion $\mathbb{L}_{\mathcal{U}}$ (Laver forcing relativized to $\mathcal{U}$) consists of trees $T \subseteq \omega^{<\omega}$ such that for each $\text{stem}(T) \subseteq s \in T : \text{succ}_T(s) := \{n \in \omega : s^\frown n \in T\} \in \mathcal{U}$, ordered by inclusion.

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Call such a $\mathcal{U}$ a *Laver ultrafilter*.

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Laver ultrafilters

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(ii) Every Laver ultrafilter is rapid.

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where $\mathcal{U} - \sum_{i\in\omega} \mathcal{V}_i$ is the ultrafilter on $\omega \times \omega$ consisting of those $A \subseteq \omega \times \omega$ satisfying $\{i \in \omega : \{j \in \omega : \langle i, j\rangle \in A\} \in \mathcal{V}_i\} \in \mathcal{U}$.

Laver ultrafilters in the Baumgartner framework

Definition (Baumgartner [1])

Let $\mathcal{I}$ be a collection of subsets of some set X (in our case, $X = 2^\omega$ or $X = \mathbb{Q} \subseteq 2^\omega$) such that $\mathcal{I}$ contains all singletons and is closed under subsets.

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Call an ultrafilter $\mathcal{U}$ over ω an *$\mathcal{I}$-ultrafilter* if for every $F : \omega \to X$, there exists some $A \in \mathcal{U}$ with $F[A] \in \mathcal{I}$.

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- the discrete subsets of 2^ω, call $\mathcal{I}$-ultrafilters *discrete ultrafilters*.

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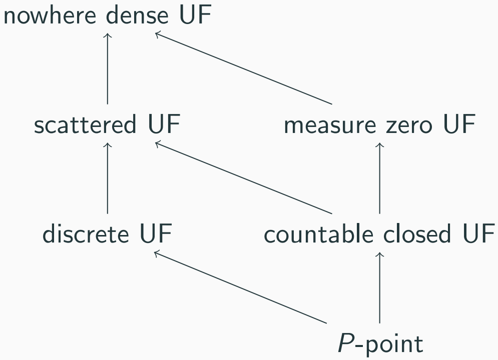
Containments among these ultrafilter classes provable in ZFC (Brendle [3]).



An arrow ($\mathcal{I}$-ultrafilter) $\to$ ($\mathcal{J}$-ultrafilter) means that every $\mathcal{I}$-ultrafilter is a $\mathcal{J}$-ultrafilter.

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$$\mathcal{I}_f := \{A \subseteq 2^\omega : \text{level}_A \leq f + 1\}.$$

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Lemma

$\mathcal{U}$ is a Laver ultrafilter if and only if $\mathcal{U}$ is an $\mathcal{I}_f$-ultrafilter for each $f \in \omega^\omega$ as above.

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Observe: Each $\mathcal{I}_f$ consists of measure zero sets, and $A \in \mathcal{I}_f \iff \bar{A} \in \mathcal{I}_f$, hence every Laver ultrafilter is a measure zero ultrafilter.

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Compare:

Theorem (Błaszczyk, Shelah [2])

$\mathbb{L}_{\mathcal{U}}$ *does not add Cohen reals if and only if* $\mathcal{U}$ *is a nowhere dense ultrafilter.*

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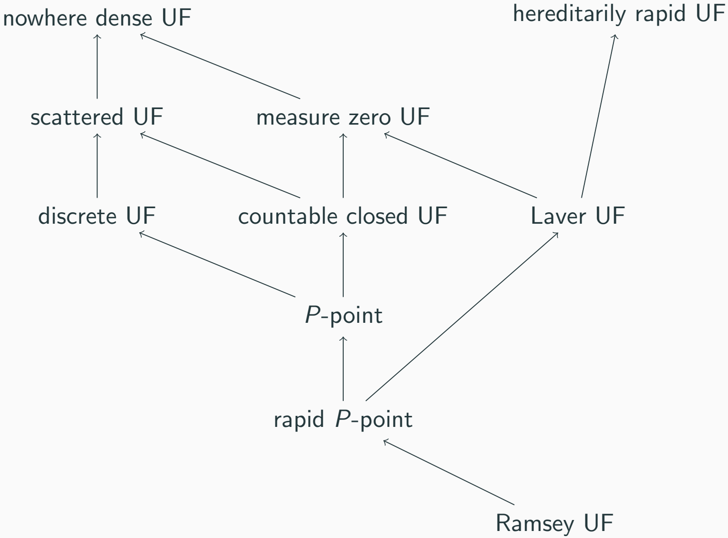
Side note: Laver ultrafilters are $\mathcal{Y}_f^0$-ultrafilters for every strictly increasing $f \in \omega^\omega$, where $\mathcal{Y}_f^0$ denotes the ideal of subsets of 2^ω with closure in the *Yorioka ideal* $\mathcal{Y}_f$ (approximations of the *strong measure zero* ideal $\mathcal{SN}$, [5]).

Extended map of ZFC-implications



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MA($\sigma -$ linked) implies the existence of a Laver ultrafilter that is not scattered.

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MA($\sigma -$ linked) implies the existence of a Laver ultrafilter that is not scattered.

Open problem: Does MA imply the existence of an ultrafilter that is both countable closed and hereditarily rapid, but not a Laver ultrafilter?

(Generic) existence of Laver ultrafilters

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An ultrafilter class $\mathcal{C}$ *exists generically* if every filter base of cardinality $< \mathfrak{c}$ can be extended to an ultrafilter in the class $\mathcal{C}$.

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(i) If $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ or $\mathrm{non}(\mathcal{NA}) = \mathfrak{c}$, then Laver ultrafilters exist generically.

(ii) If Laver ultrafilters exist generically, then $\mathrm{non}(\mathcal{SN}) = \mathfrak{c}$ and $\max\{\mathrm{non}(\mathcal{E}), \mathfrak{d}\} = \mathfrak{c}$.

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Open problem: Is the generic existence of Laver ultrafilters equivalent to $\max\{\text{cov}(\mathcal{M}), \text{non}(\mathcal{NA})\} = \mathfrak{c}$?

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Using (i):

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It is consistent that there are no P-points while Laver ultrafilters exist generically.

Proof.

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Proof.

Kill P-points using Grigorieff forcing and simultaneously increase non($\mathcal{NA}$). □

Thank you for your attention!

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