Higher Closed Null Ideal(s)

Tristan van der Vlugt (TU Wien)

joint with Adam Marton, Jaroslav Šupina and Miroslav Repický (Košice)

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Classical combinatorial set theory is the study of subsets of the reals. Usually we use ${}^{\omega}2$ or ${}^{\omega}\omega$.

The *higher context* is the result of replacing ω by a regular uncountable κ:

Classical	ω	${}^{\omega}\omega$	${}^{\omega}2$	$\mathcal{P}(\omega)$	finite	countable	$\aleph_1$
Higher	κ	${}^{\kappa}\kappa$	${}^{\kappa}2$	$\mathcal{P}(\kappa)$	$<\kappa$	$\leq\kappa$	κ^+

We will work mainly with ${}^{\kappa}2$ with the $<\kappa$-*box topology*, i.e., partial functions $s : \kappa \rightharpoonup 2$ with $|\mathrm{dom}(s)| < \kappa$ provide basic clopen sets

$$[s] = \{f \in {}^{\kappa}2 \mid s \subseteq f\}.$$

In this talk we will look at three classical σ-ideals:

- **Meagre ideal** $\mathcal{M}$:
  generated by nowhere dense subsets of ${}^\omega 2$.
- **Lebesgue null ideal** $\mathcal{N}$:
  generated by subsets of ${}^\omega 2$ of Lebesgue measure zero.
- **Closed null ideal** $\mathcal{E}$:
  generated by closed elements of $\mathcal{N}$.

Proposition

$\mathcal{E} \subseteq \mathcal{N} \cap \mathcal{M}$. $\square$

Substituting ω by κ, the κ-*meagre ideal* $\mathcal{M}_\kappa$ is the $\leq\kappa$-complete ideal generated by nowhere dense subsets of ${}^\kappa 2$.

Equivalently, $X \in \mathcal{M}_\kappa$ if and only if X is the union of κ-many nowhere dense sets.

Theorem

${}^\kappa 2$ satisfies the κ-*Baire category theorem*: if $\langle D_\alpha \mid \alpha \in \kappa\rangle$ are open dense subsets of ${}^\kappa 2$, then $\bigcap_{\alpha\in\kappa} D_\alpha$ is dense. $\square$

Equivalently, ${}^\kappa 2 \notin \mathcal{M}_\kappa$.

Since ${}^{\kappa}2$ is not second countable, we cannot simply define a *real-valued* measure that measures open subsets of ${}^{\kappa}2$.

Relaxing the notion of a Lebesgue measure to metrics defined on arbitrary monoids will not lead to a consistent generalisation of measure either (see *Agostini, Barrera, and Dimonte,* "On the problem of generalized measures: an impossibility result", 2026).

So now we wave goodbye to $\mathcal{N}$.

Higher closed null?

Since generating sets for $\mathcal{E}$ are closed, we may see them combinatorially as the body $[T]$ of trees $T \subseteq {}^{<\omega}2$.

Lemma

Let $T \subseteq {}^{<\omega}2$, then $\mu([T]) = 0$ iff for each $s \in T$ there is $n_s > \mathrm{dom}(s)$ such that for at least *half* of the extensions $t \supseteq s$ with $\mathrm{dom}(t) = n_s$ we have $t \notin T$. $\square$

Define $T \subseteq {}^{<\kappa}2$ to be *halving* if every $s \in T$ has $\alpha_s > \mathrm{dom}(s)$ such that

$$\underbrace{\left|\{t \in {}^{\alpha_s}2 \mid s \subseteq t \in T\}\right|}_{\#\text{ extensions of } s \text{ inside } T} \leq \underbrace{\left|\{t \in {}^{\alpha_s}2 \mid s \subseteq t \notin T\}\right|}_{\#\text{ extensions of } s \text{ outside } T}.$$

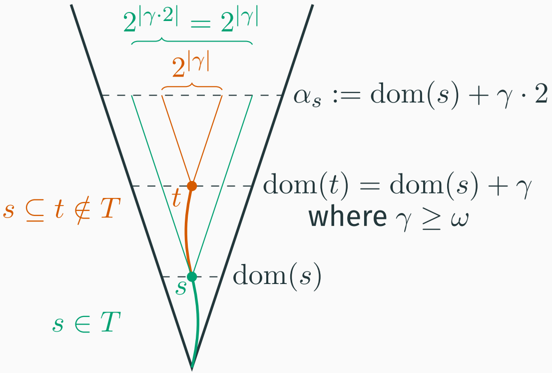
Define $\mathcal{H}_\kappa$ to be $\leq\kappa$-complete ideal generated by bodies of halving trees $T \subseteq {}^{<\kappa}2$.

Proposition

$\mathcal{H}_\kappa = \mathcal{M}_\kappa$.

Proof. Easily $[T]$ is nowhere dense for every halving $T \subseteq {}^{<\kappa}2$.

We show the reverse by picture.

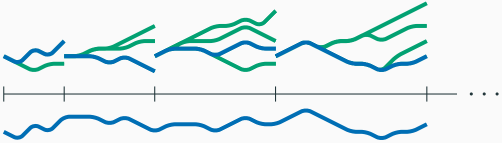


□

Characterising $\mathcal{E}$ with slaloms

Let IP denote the set of interval partitions of ω. Given $I = \langle I_n \mid n \in \omega \rangle \in \mathrm{IP}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}\left({}^{I_n}2\right)$, we write

$$[\varphi]_* = \{x \in {}^\omega 2 \mid \forall^\infty n \in \omega (x \restriction I_n \in \varphi(n))\}.$$



Theorem *Bartoszyński and Shelah 1992, Theorem 4.3*

$\mathcal{E}$ is the set of all $X \subseteq {}^\omega 2$ such that there are $\langle I_n \mid n \in \omega \rangle \in \mathrm{IP}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}\left({}^{I_n}2\right)$ with $X \subseteq [\varphi]_*$ and for which $\sum_{n\in\omega} \frac{|\varphi(n)|}{2^{|I_n|}}$ converges.

IP may be replaced by partitions of ω into finite parts.

How to deal with the convergent sum?

As mentioned, in the higher context we have no infinite summation. So, how to deal with $\sum_{n\in\omega} \frac{|\varphi(n)|}{2^{|I_n|}}$?

Theorem *Cardona, Marton, and Šupina 2025*

$X \in \mathcal{E}$ iff there are a convergent series of positive terms $\sum_{n\in\omega} \varepsilon_n$ and $I \in \mathrm{IP}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}\left({}^{I_n}2\right)$ with $X \subseteq [\varphi]_*$ and

$$\lim_{n\to\infty} \frac{|\varphi(n)|}{2^{|I_n|} \cdot \varepsilon_n} = 0.$$

We rewrite the limit as:

$$\forall m \in \omega\, \forall^\infty n \in \omega \left(|\varphi(n)| \cdot \; m \; \cdot \frac{1}{\varepsilon_n} < 2^{|I_n|} \right)$$

Now we can use that $\sum_{n\in\omega} \varepsilon_n$ converges if $\frac{1}{\varepsilon_n}$ grows *fast*.

Classical:

$X \in \mathcal{E}$ iff there are a *fast growing* $h \in {}^{\omega}\omega$ and $I \in \mathrm{IP}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}\left({}^{I_n}2\right)$ with $X \subseteq [\varphi]_*$ and

$$\forall m \in \omega\, \forall^{\infty} n \in \omega \left(|\varphi(n)| \cdot m \cdot h(n) < 2^{|I_n|} \right).$$

Higher:

We want $\lim_{\alpha\in\kappa} |I_\alpha| = \kappa$ and $|I_\alpha| < \kappa$. Hence, let κ be weakly inaccessible and IP_κ be the family of interval partitions $\langle I_\alpha \mid \alpha \in \kappa \rangle$ of κ with $\lim_{\alpha\in\kappa} |I_\alpha| = \kappa$.

$X \in \mathcal{E}_\kappa^-$ iff there are a *fast growing* $h \in {}^{\kappa}\kappa$ and $I \in \mathrm{IP}_\kappa$ and $\varphi \in \prod_{\alpha\in\kappa} \mathcal{P}\left({}^{I_\alpha}2\right)$ with $X \subseteq [\varphi]_*$ and

$$\forall \mu < \kappa\, \forall^{\infty} \alpha \in \kappa \left(|\varphi(\alpha)| \cdot \mu \cdot h(\alpha) < 2^{|I_\alpha|} \right).$$

Simplifying the higher definition

Lemma *Marton, Šupina, Repický, and vdV.*

The following are equivalent for weakly inaccessible κ:

1. $X \in \mathcal{E}_\kappa^-$
2. there are a *fast growing* $h \in {}^\kappa\kappa$ and $I \in \mathrm{IP}_\kappa$ and $\varphi \in \prod_{\alpha\in\kappa} \mathcal{P}\left({}^{I_\alpha}2\right)$ with $X \subseteq [\varphi]_*$ and
$$\forall\mu < \kappa\, \forall^\infty \alpha \in \kappa\Big(|\varphi(\alpha)| \cdot \mu \cdot h(\alpha) < 2^{|I_\alpha|}\Big),$$
3. there are $I \in \mathrm{IP}_\kappa$ and $\varphi \in \prod_{\alpha\in\kappa} \mathcal{P}\left({}^{I_\alpha}2\right)$ with $X \subseteq [\varphi]_*$ and
$$\forall^\infty \alpha \in \kappa\Big(|\varphi(\alpha)| < 2^{|I_\alpha|}\Big).$$

Let BP_κ be the set of partitions $\langle B_\alpha \mid \alpha \in \kappa \rangle$ of κ such that $B_\alpha \in [\kappa]^{<\kappa}$ and $\lim_{\alpha \in \kappa} |B_\alpha| = \kappa$.

Define $\mathcal{BE}_\kappa^-$ as the family of sets $X \subseteq {}^\kappa 2$ such that there are $B \in \mathrm{BP}_\kappa$ and $\varphi \in \prod_{\alpha \in \kappa} \mathcal{P}\left({}^{B_\alpha} 2\right)$ with $X \subseteq [\varphi]_*$ and

$$\forall^\infty \alpha \in \kappa \Big(|\varphi(\alpha)| < 2^{|B_\alpha|} \Big).$$

Theorem *Marton, Šupina, Repický, and vdV.*

For weakly inaccessible κ, we have:

$$\mathcal{E}_\kappa^- \subsetneq \mathcal{BE}_\kappa^- \subsetneq \mathcal{M}_\kappa.$$

Not ideal

One may ask if $\mathcal{E}_\kappa^- = \mathcal{E}_\kappa$ and $\mathcal{B}\mathcal{E}_\kappa^- = \mathcal{B}\mathcal{E}_\kappa$.
This turns out to be **false**.

Theorem *Marton, Šupina, Repický, and vdV.*
If κ is weakly inaccessible, then there are $X, Y \in \mathcal{B}\mathcal{E}_\kappa^-$ such that $X \cup Y \notin \mathcal{B}\mathcal{E}_\kappa^-$.

If κ is strongly inaccessible, then there are $X, Y \in \mathcal{E}_\kappa^-$ such that $X \cup Y \notin \mathcal{E}_\kappa^-$.

Let $\mathcal{E}_\kappa$ and $\mathcal{B}\mathcal{E}_\kappa$ be the $\leq\kappa$-complete ideals generated by $\mathcal{E}_\kappa^-$ and $\mathcal{B}\mathcal{E}_\kappa^-$, respectively. Then:

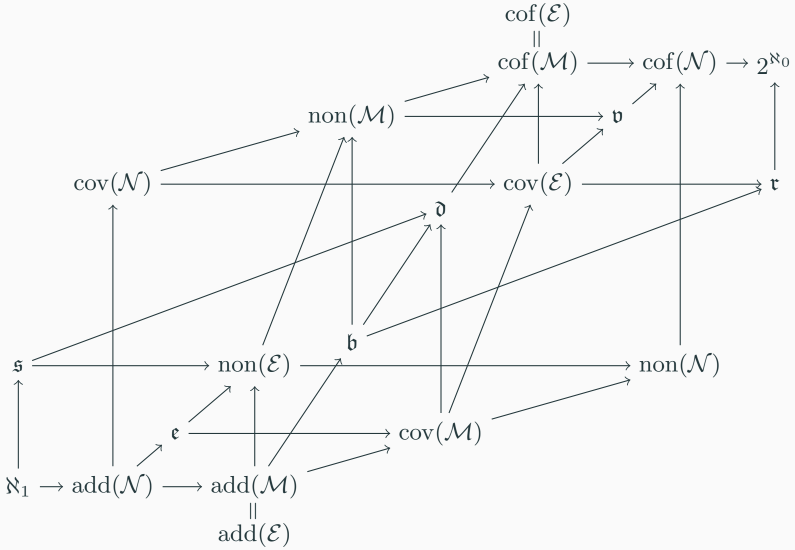
$$\mathcal{E}_\kappa \subsetneq \mathcal{B}\mathcal{E}_\kappa \subsetneq \mathcal{M}_\kappa.$$

Overview of families

$$
\begin{array}{ccccccc}
\mathcal{E}_\kappa & \longrightarrow & \mathcal{BE}_\kappa & \longrightarrow & \mathcal{M}_\kappa & \longrightarrow & \mathcal{P}({}^\kappa 2) \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{E}_\kappa^- & \longrightarrow & \mathcal{BE}_\kappa^- & & \mathcal{NWD}_\kappa & &
\end{array}
$$

All inclusions are strict.

Classical cardinal characteristics



The localisation number

Let $\varphi \in \prod_{\alpha\in\kappa}[\kappa]^{|\alpha|}$ and $f \in {}^\kappa\kappa$, then we say φ *localises* f if $f(\alpha) \in \varphi(\alpha)$ for almost all $\alpha \in \kappa$.

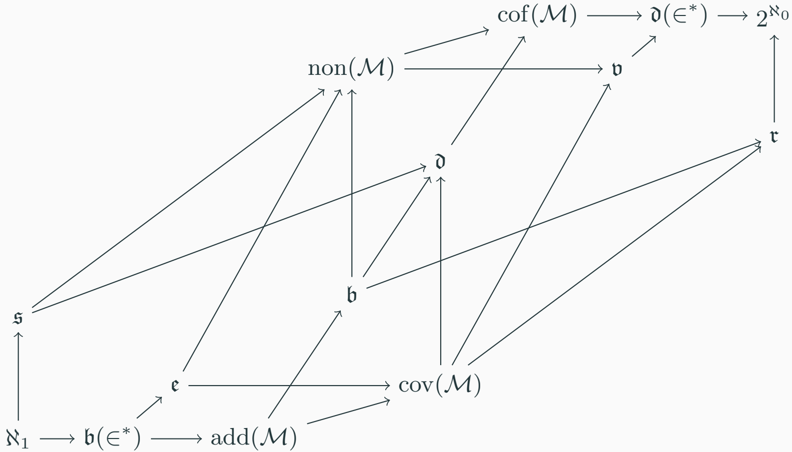
A family $L \subseteq \prod_{\alpha\in\kappa}[\kappa]^{|\alpha|}$ is called *localising* if every $f \in {}^\kappa\kappa$ is localised by some $\varphi \in L$, and let us call a family $U \subseteq {}^\kappa\kappa$ *unlocalisable* if no $\varphi \in \prod_{\alpha\in\kappa}[\kappa]^{|\alpha|}$ localises all $f \in U$.

The *localisation* and *unlocalisation* numbers $\mathfrak{d}_\kappa(\in^*)$ and $\mathfrak{b}_\kappa(\in^*)$ are the least size of localising and unlocalisable families, respectively.

Theorem *Bartoszyński 1987*

$\mathrm{cof}(\mathcal{N}) = \mathfrak{d}(\in^*)$ and $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\in^*)$. $\square$

From classical to higher



From classical to higher

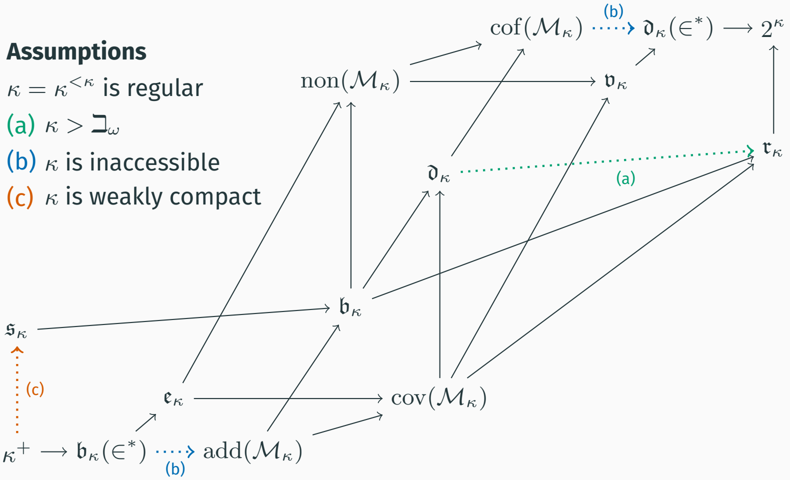
Assumptions

$\kappa = \kappa^{<\kappa}$ is regular

(a) $\kappa > \beth_\omega$

(b) κ is inaccessible

(c) κ is weakly compact



We have the following ZFC-provable results for κ strongly inaccessible:

Classical	$\mathcal{E}_\kappa$	$\mathcal{BE}_\kappa$
$\mathrm{non}(\mathcal{E}) \leq \mathrm{non}(\mathcal{M})$	$\mathrm{non}(\mathcal{E}_\kappa) \leq \mathrm{non}(\mathcal{BE}_\kappa) \leq \mathrm{non}(\mathcal{M}_\kappa)$	
$\mathrm{add}(\mathcal{E}) \leq \mathfrak{b}$	$\mathrm{add}(\mathcal{E}_\kappa) \leq \mathfrak{b}_\kappa$	?
$\mathfrak{s} \leq \mathrm{non}(\mathcal{E})$	?	$\mathfrak{s}_\kappa \leq \mathrm{non}(\mathcal{BE}_\kappa)$
$\mathfrak{e} \leq \mathrm{non}(\mathcal{E})$	?	$\mathfrak{e}_\kappa \leq \mathrm{non}(\mathcal{BE}_\kappa)$
$\mathfrak{b}(\in^*) \leq \mathrm{add}(\mathcal{E})$	$\mathfrak{b}_\kappa(\in^*) \leq \mathrm{non}(\mathcal{E}_\kappa)$	$\mathfrak{b}_\kappa(\in^*) \leq \mathrm{non}(\mathcal{BE}_\kappa)$
$\mathrm{add}(\mathcal{M}) \leq \mathrm{non}(\mathcal{E})$	?	?
$\mathrm{add}(\mathcal{M}) \leq \mathrm{add}(\mathcal{E})$	?	?
$\mathrm{add}(\mathcal{E}) \leq \mathrm{cov}(\mathcal{M})$	?	?

Those results that are known also dualise.

For $f, g \in {}^\kappa\kappa$, define $f \leq^{\mathrm{cl}} g$ if there is a club set $C \subseteq \kappa$ such that $f(\alpha) \leq g(\alpha)$ for all $\alpha \in C$. We define $\mathfrak{b}_\kappa^{\mathrm{cl}}$ as the least size of a $\leq^{\mathrm{cl}}$-unbounded family.

Theorem *Cummings and Shelah 1995, Theorem 6*

$\mathfrak{b}_\kappa^{\mathrm{cl}} = \mathfrak{b}_\kappa$.

We may dually describe $\mathfrak{d}_\kappa^{\mathrm{cl}}$ as the least size of a $\leq^{\mathrm{cl}}$-dominating family. There is the following long-standing open question:

Theorem *Cummings and Shelah 1995*

Is $\mathfrak{d}_\kappa^{\mathrm{cl}} = \mathfrak{d}_\kappa$?

Theorem *Cummings and Shelah 1995, Theorem 8*

If $\kappa > \beth_\omega$, then $\mathfrak{d}_\kappa^{\mathrm{cl}} = \mathfrak{d}_\kappa$. (E.g. κ strongly inaccessible)

Theorem *Marton, Šupina, Repický, and vdV.*

$\mathrm{add}(\mathcal{E}_\kappa) \leq \mathfrak{b}^{\mathrm{cl}}_\kappa$.

Main point of the proof. If $I \in \mathrm{IP}_\kappa$, then $\{\min(I_\alpha) \mid \alpha \in \kappa\}$ is a club set.

Hence, if $\bigcup_{\xi\in\kappa}[\psi^\xi]_* \in \mathcal{E}_\kappa$, where each slalom ψ^ξ has associated partition $J^\xi \in \mathrm{IP}_\kappa$, then there is a club C such that C is almost contained in $\left\{ \min(J^\xi_\alpha) \mid \alpha \in \kappa \right\}$ for each ξ.

To finish a proof, one constructs slaloms φ_b for each $b \in B$, where B is a $\leq^{\mathrm{cl}}$-unbounded. Then show that

$$[\varphi_b]_* \not\subseteq \bigcup_{\xi\in\kappa}[\psi^\xi]_*$$

for one of those $b \in B$.

This does not work for $\mathcal{B}\mathcal{E}_\kappa$ ...

Theorem *Marton, Šupina, Repický, and vdV.*

$\mathfrak{s}_\kappa \leq \mathrm{non}(\mathcal{BE}_\kappa)$.

Proof. If $A \subseteq {}^\kappa 2$ and $A \notin \mathcal{BE}_\kappa$, then $\{x^{-1}(1) \mid x \in A\}$ is a splitting family.

Let $Y \in [\kappa]^\kappa$ and without loss we assume $|\kappa \setminus Y| = \kappa$, then we define $B \in \mathrm{BP}_\kappa$ such that $|B_\alpha \cap Y| = 2^{|\alpha|}$ and $|B_\alpha \setminus Y| = |\alpha|$. Define $\varphi \in \prod \mathcal{P}\left({}^{B_\alpha}2\right)$ by

$$\varphi(\alpha) = \left\{s \in {}^{B_\alpha}2 \;\middle|\; s \restriction (B_\alpha \cap Y) \text{ is constant}\right\}$$

Since $A \notin \mathcal{BE}_\kappa$, there is $x \in A$ with $x \notin [\varphi]_*$. Hence, for cofinally many $\alpha \in \kappa$ we see that $x \restriction (B_\alpha \cap Y)$ is not constant. It follows that $\left|\{x(\xi) = i \mid \xi \in Y\}\right| = \kappa$ for both $i \in 2$. $\square$

This does not work for $\mathcal{E}_\kappa$ ...


Thank you!
Děkuji za pozornost!