Groupwise dense number after Combinatorial tree forcing of meager ideals

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Outline

- **Main Objective**:
  - Investigate **cardinal invariants** after forcing with the Borel σ-algebra modulo **the σ-ideal generated by closed sets of null ideal** on 2^ω.

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Note that the forcing of Borel σ-algebra modulo **the σ-ideal generated by closed null sets on** 2^ω is represented by a combinatorial tree forcing on a **meager ideal on** ω.

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Theorem 2 (Main Theorem)

Assume that J *is a* ***meager*** *ideal on* ω. *Then, the countable support iteration of* ***combinatorial tree forcing*** $Q(J)$ *of length* ω_2 *forces that* $\mathfrak{g} = \aleph_2$.

Definition 3 (Groupwise dense set)

A set $G \subseteq [\omega]^\omega$ is **groupwise dense** if

1. For $a, b \in [\omega]^\omega$, if $a \in G$ and $b \subseteq^* a$, then $b \in G$.
2. For any interval partition $\langle I_n : n \in \omega \rangle$ of ω, there is some $A \in [\omega]^\omega$ such that $\bigcup_{n \in A} I_n \in G$.

Definitions and Preliminaries

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Definition 4 (Groupwise dense number)

$\mathfrak{g} := \min\{\kappa : \exists \langle G_\alpha : \alpha < \kappa \rangle$ groupwise dense sets, $\bigcap_{\alpha<\kappa} G_\alpha = \emptyset\}$.

Definition 5 (Combinatorial tree forcing)

Combinatorilal tree forcing $Q(J)$ consists of those trees $T \subseteq \omega^{<\omega}$ for which every node $t \in T$ has an extension $s \in T$ satisfying $succ_T(s) \notin J$. $Q(J)$ is ordered by inclusion.

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Basic Property of a meager ideal on ω (Jalani-Naini, Talagrand)

An ideal J on ω is **meager** iff:
we can find a finite interval partition $\langle I_\ell : \ell < \omega \rangle$ of ω such that for any infinite $A \subseteq \omega$, $\bigcup_{\ell \in A} I_\ell \notin J$.

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Lemma 7 (Key Lemma)

*Assume that J is a **meager** ideal on ω.*
*Let G be a **groupwise dense** set in the ground model V, $\dot{r} \in \omega^\omega$ be a name for the **generic real** of $\mathbb{Q}(J)$, and $\mathrm{ran}^*(\dot{r})$ denotes the name of the set $\{\dot{r}(j) : \forall i < j, \dot{r}(i) < \dot{r}(j)\}$.*

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Lemma 7 (Key Lemma)

*Assume that J is a **meager** ideal on ω.*
*Let G be a **groupwise dense** set in the ground model V, $\dot{r} \in \omega^\omega$ be a name for the **generic real** of $Q(J)$, and $ran^*(\dot{r})$ denotes the name of the set $\{\dot{r}(j) : \forall i < j, \dot{r}(i) < \dot{r}(j)\}$.*
Then $Q(J)$ forces that $\exists b \in G, ran^*(\dot{r}) \subseteq^* b$.

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If $n \notin B$, **chop off the nodes** in L_n^s from $\langle L_n^s : n < \omega \rangle$ for $s \in \bigcup_{x<\omega} S_x$.

Finally, the pruned condition forces that $ran^*(\dot{r}) \subseteq^* \bigcup_{i \in B} [k_i, k_{i+1}) \in G$. □

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Question:

- Combinatorial tree forcings of **non-meager** ideals also increase the **groupwise dense number**?

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Thank you for your attention!

References

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