The Closed Subtree Property for Aronszajn trees

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Summary about Aronszajn trees

In 1920, Suslin posed the Suslin Hypothesis. The Suslin Hypothesis is equivalent to the statement:

"There is no ω_1-tree without uncountable chains or antichains (i.e., no Suslin tree exists)."

- Aronszajn first constructed an ω_1-tree without uncountable chains in ZFC (i.e., Aronszajn trees exist in ZFC).
- The existence of a Suslin trees is consistent with ZFC.
- The Suslin Hypothesis is also consistent with ZFC.

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Background and Moore's results

Woodin's Problem

Are there Π_2-sentences ψ_1 and ψ_2 such that

$$(H(\aleph_2), \in) \models \mathsf{CH} \wedge \psi_1 \text{ and}$$
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Theorem (Moore 2007 [1])

If there is an Aronszajn tree T with the following conditions, then $2^{\aleph_0} = 2^{\aleph_1}$.

1. *T is club-embeddable into all of its uncountable subtrees.*
2. *Every ladder system coloring can be T-uniformized.*

Woodin's problem was resolved in 2013 by Asperó-Larson-Moore [2].

Definition (Moore 2005)

$(T, <_T)$: Aronszajn tree, $W \subseteq T$: uncountable subtree.

- W is T-closed if every T-bounded increasing sequence of W has an upper bound in W.
- T has the closed subtree property if every uncountable subtree W of T has an uncountable T-closed subtree of W.

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- $\sigma_p \in [W]^{<\aleph_0}$, $\mathrm{dom}(f_p) \in [T_{\mathsf{Lim}}]^{<\aleph_0}$, $f_p \colon \mathrm{dom}(f_p) \to T_{\mathsf{Succ}}$, and
- $\forall t \in \mathrm{dom}(f_p)(f_p(t) <_T t \wedge (\sigma_p{\downarrow_T}) \cap [f_p(t), t]_T = \emptyset)$.

$q \leq_P p$ **iff** $\sigma_q \supseteq \sigma_p$ and $f_q \supseteq f_p$.

Notation

- T_{Lim} is the set of all limit height elements of T.
- T_{Succ} is the set of all successor height elements of T.
- $\sigma_p{\downarrow_T} := \{x \in T \mid \exists s \in \sigma_p(x \leq_T s)\}$.
- $[f_p(t), t]_T := \{x \in T \mid f_p(t) \leq_T x \leq_T t\}$.

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Proof of Claim 2 Define the subtree $X := \bigcup\{\sigma_p{\downarrow_T} \mid p \in G\}$.

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Claim 2 A generic filter G adds T-closed subtree of W.

Proof of Claim 2 Define the subtree $X := \bigcup\{\sigma_p{\downarrow_T} \mid p \in G\}$. By Claim 1, for every $t \in T_{\mathsf{Lim}}$, either $t \in X$ or $\exists p \in G([f_p(t), t] \cap X = \emptyset)$. Hence, X is a T-closed subtree of W.

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Claim 2 A generic filter G adds T-closed subtree of W.

Claim 3 P has the countable chain condition. $\square$

Aronszajn trees with/without the closed subtree property

Natural Question

What relationship do the csp Aronszajn trees have with other Aronszajn trees? (e.g., Special Aronszajn trees, Suslin trees)

Proposition

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Proof. Let S be a Suslin tree and W an uncountable subtree of S. We may assume that W is well-prouned. For every $t \in S \setminus W$, we define $\alpha_t := \min\{\alpha \leq \mathsf{ht}(t) \mid t \restriction \alpha \notin W\}$.

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Then $X := \{x \in S \mid x \not<_S w\}$ is uncountable and S-closed.

$(\because)\ \forall x \in S \setminus W(\mathsf{ht}(x) \geq \delta \to x \restriction \delta \notin W)$ □

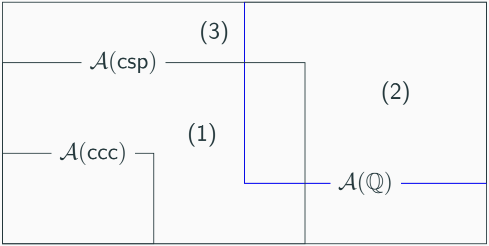
Theorem

- $\mathcal{A}(\mathrm{ccc}) \subseteq \mathcal{A}(\mathrm{csp})$.

(1) *If a Suslin tree exists, then* $\mathcal{A}(\mathrm{csp}) \setminus (\mathcal{A}(\mathbb{Q}) \cup \mathcal{A}(\mathrm{ccc})) \neq \emptyset$.

(2) *If* $\diamondsuit$ *holds, then* $\mathcal{A}(\mathbb{Q}) \setminus \mathcal{A}(\mathrm{csp}) \neq \emptyset$.

(3) *If* $\diamondsuit$ *holds, then* $\mathcal{A} \setminus (\mathcal{A}(\mathbb{Q}) \cup \mathcal{A}(\mathrm{csp})) \neq \emptyset$.



Some restricted forms of Martin's Axiom

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Definition

(Larson-Todorčević [3], Yorioka [4, Definition 2.4])

Let P be a forcing notion. P has rec if there exists an assignment π from P into $[\omega_1]^{<\aleph_0}$ with $[\mathbf{com}]$ and $[\mathbf{rec}]$.

$[\mathbf{com}]$ For any pair of compatible conditions $p, q \in P$, there is a common extension r of p and q in P such that $\pi(r) = \pi(p) \cup \pi(q)$.

$[\mathbf{rec}]$ For any pair of disjoint uncountable subsets I and J of P, if $\{\pi(p) \mid p \in I \cup J\}$ forms a Δ-system, then there are uncountable subsets I' and J' of I and J, respectively, such that any member of I' is compatible with any member of J' in P.

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Definition **(Yorioka [4, Definition 2.4])**

Let P be a forcing notion. P has $\mathsf{R}_{1,\aleph_1}$ if there exists an assignment π from P into $[\omega_1]^{<\aleph_0}$ with $[\mathbf{com}]$ and $[\mathbf{R_{1,\aleph_1}}]$.

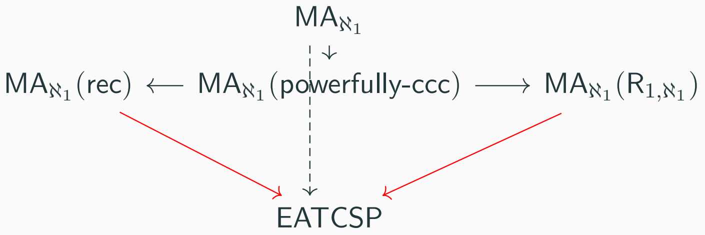
$[\mathbf{com}]$ For any pair of compatible conditions $p, q \in P$, there is a common extension r of p and q in P such that $\pi(r) = \pi(p) \cup \pi(q)$.

$[\mathbf{R_{1,\aleph_1}}]$ For any uncountable subset I of P, for any countable elementary submodel N of $H(\theta)$ with $\{P, \pi, I\} \in N$ and for any $p \in P \setminus N$, if $\{\pi(q) \mid q \in I\}$ forms a Δ-system with root $\pi(p) \cap N$, then there is an uncounatble subset I' of I such that $I' \in N$ and any member of I' is compatible with p in P.

Given an Aronszajn tree T and its uncountable subtree W, Moore found a ccc forcing notion $P_{T,W}$ such that it adds an uncountable T-closed subtree of W in an extension.

Proposition

The forcing notion $P_{T,W}$ defined by Moore actually has two properties rec *and* $\mathsf{R}_{1,\aleph_1}$ *which are stronger than the countable chain condition.*



Proposition

If Suslin Axiom holds, then the Suslin tree forces that every Aronszajn tree has the closed subtree property.

Y. Aoki introduced a forcing property named the eventual precaliber $\aleph_1$ ($\mathsf{EPC}^*_{\aleph_1}$) in his 2024 paper [5].

Definition **(N.)**

A forcing notion P has wE*-ccc if whenever $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ is an uncountable sequence of P, there exists an uncountable subset I of ω_1 and a sequenece $\langle \tilde{p}_\alpha \mid \alpha \in I \rangle$ such that each $\tilde{p}_\alpha$ is an extension of p_α, and the following holds:

$$\exists p^* \in P \forall p \leq_P p^* \forall J \in [I]^{\aleph_1} \exists \alpha \in J (\tilde{p}_\alpha \not\perp_P p).$$

- Every wE*-ccc forcing notion has powerfully-ccc. In particular, the forcing notion has ccc.
- Every finite support iteration of wE*-ccc forcings has wE*-ccc.

Theorem

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If such a node does not exists, then the following set becomes an uncountable T-closed subtree of W:

$$\{s \in T \mid \forall \xi < \mathrm{ht}_T(s)(|\{\alpha \in I \mid s \restriction \xi \leq_T t_\alpha\}| = \aleph_1)\}.$$

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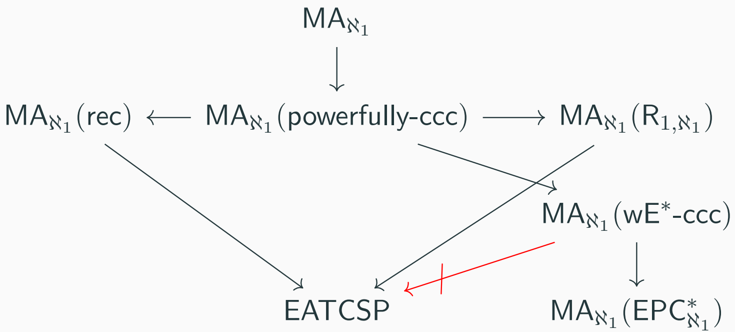
By $(\star)$, $p^* \Vdash_P$ "$\forall \xi < \mathrm{ht}_T(t)(t \restriction \xi \in \dot{X})$". And, $p^* \Vdash_P$ "$t \notin \dot{X} \subseteq W$" because $t \notin W$. □

Theorem

Every wE*-ccc *forcing preserve an Aronszajn tree.*



Open question

Question

Is there a property φ for Aronszajn trees such that the following hold?

- It is consistent that every Aronszajn tree has the property φ.
- It is consistent that every Aronszajn tree has the property $\neg\varphi$.

- The speciality on Aromszajn trees satisfies the first statement, but fails the second statement.
- Conversely, the countable chain condition on Aronszajn trees satisfies the second statement, but fails the first statement.

[1] J. T. Moore.
ω_1 and $-\omega_1$ may be the only minimal uncountable linear orders.
Michigan Math. J., 55(2):437–457, 2007.

[2] D. Asperó, P. Larson, and J. T. Moore.
Forcing axioms and the continuum hypothesis.
Acta Math., 210(1):1–29, 2013.

[3] P. Larson and S. Todorčević.
Katětov's problem.
Trans. Amer. Math. Soc., 354(5):1783–1791, 2002.

[4] T. Yorioka.
Two chain conditions and their Todorčević's fragments of Martin's axiom.
Ann. Pure Appl. Logic, 175(1):Paper No. 103320, 28, 2024.

[5] Y. Aoki.
Discontinuous homomorphisms on $C(X)$ with the negation of CH and a weak forcing axiom.
J. Lond. Math. Soc. (2), 110(1):Paper No. e12956, 39, 2024.

Thank you for your attention