On uniformly Menger elements

Mbekezeli Nxumalo

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Preliminaries

- A *poset* is a pair (L, R) where L is a set, and $R \subseteq L \times L$ is a binary relation on L satisfying:
  1. $\forall a \in L, aRa$.
  2. $\forall a, b, c \in L$, aRb and $bRc \implies aRc$.
  3. $\forall a, b \in L$, aRb and $bRa \implies a = b$.

  When these three properties hold, R is called a *(partial) order* on L.
- If there is no danger of confusion we typically use for an order the symbol $\leq$, even for distinct relations on distinct sets.
- For a poset $(L, \leq)$, we shall write L if $\leq$ is clear from the context.
- The *supremum* of a set $M \subseteq (L \leq)$, denoted by $\sup M$, is the least upperbound of M. Similarly, the *infimum* of M, denoted by $\inf M$, is the greatest lowerbound of M.

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- For a poset L, we write $\bigvee M$ and $\bigwedge M$ for the supremum and the infimum of $M \subseteq L$, respectively. For $\{a, b\} \subseteq L$, we write $a \wedge b$ and $a \vee b$.
- Given a poset L, we have that $\sup \emptyset$ is the least element of L and we denote it by 0_L and shall be called the *bottom* element of L. Similarly, $\inf \emptyset$ is the greatest element of L and we denote it by 1_L, and shall be called the *top* element of L.
- A poset L is a *lattice* if there is an infimum $a \wedge b$ and a supremum $a \vee b$ for any two $a, b \in L$. It is a *bounded lattice* if it is a lattice that has bottom and top.
- A lattice L is distributive if there holds the equality:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

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- A poset is a *complete lattice* if every subset has a supremum and an infimum.

Proposition 1.

Let L be a poset. If each subset of L has a supremum, then L is a complete lattice.

- Order preserving maps $f : L \to M$ and $g : M \to L$ are *Galois adjoint - f is a left adjoint of g, and g is a right adjoint of f-* if

$$\forall a \in L, \forall b \in M, \quad f(a) \leq b \iff a \leq g(b).$$

- A left (resp. right) Galois adjoint of a given map does not have to exist. If it exists, however, it is uniquely determined.

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Proposition 2.

If L and M are complete lattices, then an order preserving map $f : L \to M$ is a left (resp. right) adjoint if and only if it preserves all suprema (resp. infima).

- In a distributive lattice L, a *pseudocomplement* of $a \in L$ is the largest element $b \in L$ such that $a \wedge b = 0$, if it exists.
- We denote the pseudocomplement of $a \in L$ by a^*. It has a property that

$$a \wedge b = 0 \iff b \leq a^*$$

- A *Heyting algebra* is a bounded lattice L equipped with a binary operation $\to$ satisfying

$$c \leq a \to b \quad \iff \quad c \wedge a \leq b.$$

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Proposition 3.

A complete lattice L admits a Heyting operation iff there holds the distributive law:

$$a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b)$$

for all $a \in L$, $B \subseteq L$.

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- A *frame* or a *locale* is a complete lattice L satisfying the following infinite distributive property:

$$a \wedge \bigvee B = \bigvee_{b \in B} (a \wedge b)$$

for all $a \in L$, $B \subseteq L$.

Example 1.

Let (X, τ) be a topological space. Then τ is a frame, where $\wedge = \cap$, and $\bigvee = \bigcup$.

- Given a space X, we set $\mathfrak{O}X = \{\text{open sets of } X\}$ and call $\mathfrak{O}X$ the *frame of opens* of X.
- A *frame homomorphism* is a function $h : L \to M$ which preserves arbitrary joins (including the bottom element) and finite meets (including the top element).

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Example 2.

Let $f : X \to Y$ be a continuous function between spaces X and Y. Then the map

$$\mathfrak{O}(f) : \mathfrak{O}Y \to \mathfrak{O}X, \quad \mathfrak{O}(f)(U) = f^{-1}[U]$$

is a frame homomorphism.

- Associated with any frame homomorphism $h : L \to M$ is a right adjoint $h_* : M \to L$ which is called a *localic map*.

Example 3.

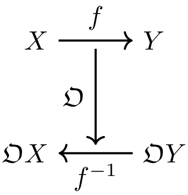
Let $f : X \to Y$ be a continuous function between spaces X and Y. Then the map

$$\mathrm{Lc}(f) : \mathfrak{O}X \to \mathfrak{O}Y, \quad \mathrm{Lc}(f)(U) = \mathfrak{O}(f)_*(U) = Y \smallsetminus \overline{f[X \smallsetminus U]}$$

is a localic map.

Preliminaries

- Denote by:
  - **Top** the category of topological spaces whose morphisms are continuous functions.
  - **Frm** the category of frames whose morphisms are frame homomorphisms.
  - **Loc** the category of locales whose morphisms are localic maps.
- There is a contravariant functor $\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}$ which acts as follows:



Preliminaries

- A *sublocale* of a frame L is a subset $S \subseteq L$ satisfying:
  - $\bigwedge^S = \bigwedge^L$
  - For each $x \in L$ and each $s \in S$, $x \to s \in S$.
- $\mathcal{S}(L)$ denotes the collection of all sublocales of a frame L.

Example 4.

Let X be a space. For each $A \subseteq X$, set

$$\widetilde{A} = \{\mathrm{int}((X \smallsetminus A) \cup W) : W \in \mathfrak{O}X\}.$$

$\widetilde{A}$ is a sublocale of $\mathfrak{O}X$ and is called *the sublocale induced by* A.

- In a T_D-space X, $A \subseteq B$ if and only if $\widetilde{A} \subseteq \widetilde{B}$.
- Given $a \in L$, we denote by $\mathfrak{c}(a) = \{x \in L : a \leq x\}$ the *closed sublocale of* L *induced by* a and by $\mathfrak{o}(a) = \{a \to x : x \in L\}$ the *open sublocale of* L *induced by* a.

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Example 5.

Let X be a space and let $U \in \mathfrak{O}X$. Then

$$\mathfrak{o}(U) = \widetilde{U}$$

and

$$\mathfrak{c}(U) = \uparrow U.$$

- By a *cover* of a frame L we refer to $C \subseteq L$ in which $\bigvee C = 1$. Denote by $\mathrm{Cov}(L)$ the collection of all covers of a frame L.
- A collection $\mathcal{C} \subseteq \mathcal{S}(L)$ is a *covering* of L if $\bigvee\{C : C \in \mathcal{C}\} = L$, where the join is calculated in $\mathcal{S}(L)$.
- Let L be a frame and $A, B \in \mathrm{Cov}(L)$. Then $A \leq B$ in case for each $a \in A$, there is $b \in B$ such that $a \leq b$.
- For $A, B \in \mathrm{Cov}(L)$, we write $A \wedge B = \{a \wedge b : a \in A, b \in B\}$.

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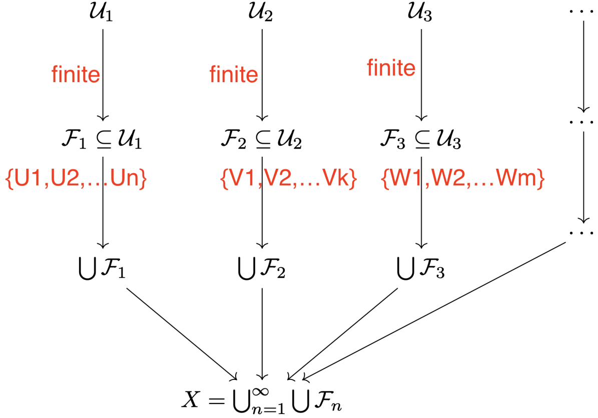
- For $A \in \mathrm{Cov}(L)$, set $Ax = \bigvee\{t \in A : t \wedge x \neq 0\}$.
- For $\mu \subseteq \mathrm{Cov}(L)$, we write $a \lhd_\mu b$ if there exists $A \in \mu$ such that $Aa \leq b$.
- Let L be a frame. A collection $\mu \subseteq \mathrm{Cov}(L)$ is called a *nearness* on L if:
  1. Whenever $A \in \mathrm{Cov}(L)$ and $B \in \mu$ with $B \leq A$, we have $A \in \mu$;
  2. $A, B \in \mu$ implies $A \wedge B \in \mu$;
  3. For each $x \in L$,
  $$x = \bigvee\{y \in L : y \lhd_\mu x\}.$$
- The pair (L, μ), where μ is a nearness on L, is called a *nearness frame*, and members of μ are called *uniform covers*.

Introducing uniformly Menger elements

- **Menger space**: A space X is *Menger* if for every sequence $\{\mathcal{C}_n : n \in \mathbb{N}\}$ of open covers of X we can select, for each n, a finite set $\mathcal{V}_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a cover of X.



Introducing uniformly Menger elements

- **Menger frame**: (Bayih, Dube, Ighedo 2021) In pointfree topology, a frame L is *Menger* if for every sequence $(\mathcal{C}_n)$ of open coverings of L, there exists, for each n, a finite $\mathcal{D}_n \subseteq \mathcal{C}_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n$ is a covering of L.

Proposition 4.

Let X be a space. Then X is a Menger space if and only if $\mathfrak{O}X$ is a Menger frame.

- **Relatively Menger subset**: (Sen 2023) A subset A of a space X is *relatively Menger* if for every sequence $\{\mathcal{C}_n : n \in \mathbb{N}\}$ of open covers of X, there exists, for each n, a finite set $J_n \subseteq C_n$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} J_n$.

Introducing uniformly Menger elements

- **Relatively Menger sublocale**: In frames, a sublocale S of a frame L is *relatively Menger* if for every sequence $\{\mathcal{C}_n : n \in \mathbb{N}\}$ of open coverings of L, there exists, for each n, a finite set $\mathcal{D}_n \subseteq \mathcal{C}_n$ such that $S \subseteq \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{D}_n$.

Proposition 5.

Let X be a T_D-space and $S \subseteq X$. Then S is a relatively Menger subset of X if and only if $\widetilde{S}$ is a relatively Menger subublocale of $\mathfrak{O}X$.

Introducing uniformly Menger elements

- **Menger element**: An element a of a frame L is *Menger* if for every countable collection $\{C_n : n \in \mathbb{N}\}$ of covers of L, there exists, for each n, a finite set $D_n \subseteq C_n$ such that $a \leq \bigvee_{n\in\mathbb{N}} (\bigvee D_n)$.
- **Uniform Menger property**: Let $(X, \mathcal{U})$ be a uniform space. Then X has the *uniform Menger property* if for every sequence $\{\mathcal{C}_n : n \in \mathbb{N}\}$ of uniform covers of X there exists, for each n, a finite set $\mathcal{V}_n \subseteq C_n$ such that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ covers X.

Definition 6.

Let (L, μ) be a nearness frame. An element $a \in L$ is *uniformly Menger* if for every countable collection $\{C_n : n \in \mathbb{N}\}$ of uniform covers of L, there exists, for each n, a finite set $D_n \subseteq C_n$ such that $a \leq \bigvee_{n\in\mathbb{N}} (\bigvee D_n)$.

References

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Thank you.

Ngiyabonga.

Ndiyabulela.

Dankie.

Kea leboha.

Ngiyathokoza.

Inkomu.

Ke a leboga.

Ndo livhuwa.

Děkuju.