A topological characterization of $\min\{\mathfrak{r}, \mathfrak{d}\}$

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Joint work with Roman Pol and Piotr Zakrzewski

A result of Sierpiński, starting point

Theorem (Sierpinski 1934)

(CH) There exists an uncountable $X \subset [0,1]$ *and continuous* $f : X \to [0,1]$ *such that* $f \restriction Y$ *is not uniformly continuous for any uncountable* $Y \subset X$. □

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There exists a monotone function $f_S : [0,1] \to [0,1]$ such that for any $\kappa \leq \mathfrak{c}$ and κ-Lusin set $X \subset [0,1]$, $f_S \restriction Y$ is not uniformly continuous on Y for any $Y \in [X]^\kappa$.

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Remark. The proof of the first item heavily used that $|\overline{G(h_{KS})} \cap (\{q\} \times [0,1])| = \mathfrak{c}$ for all $q \in \mathbb{Q} \cap (0,1)$, which in a sense says that h_{KS} is "very non-monotone",

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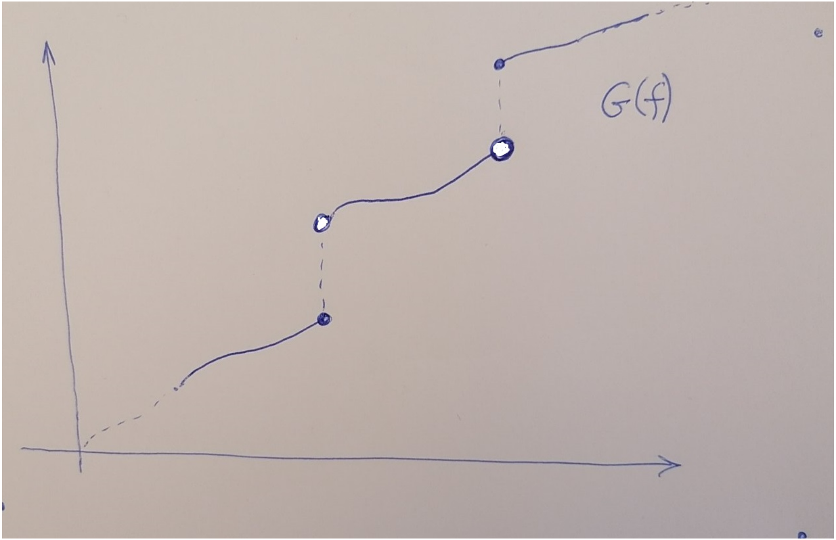

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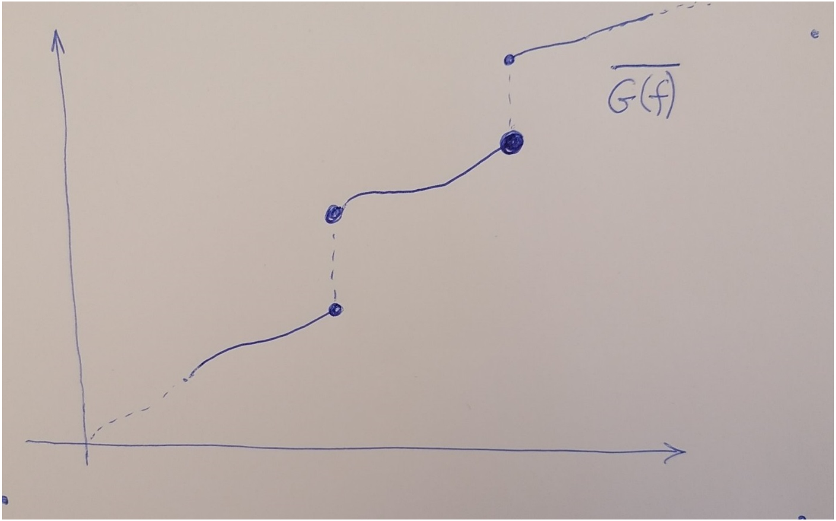

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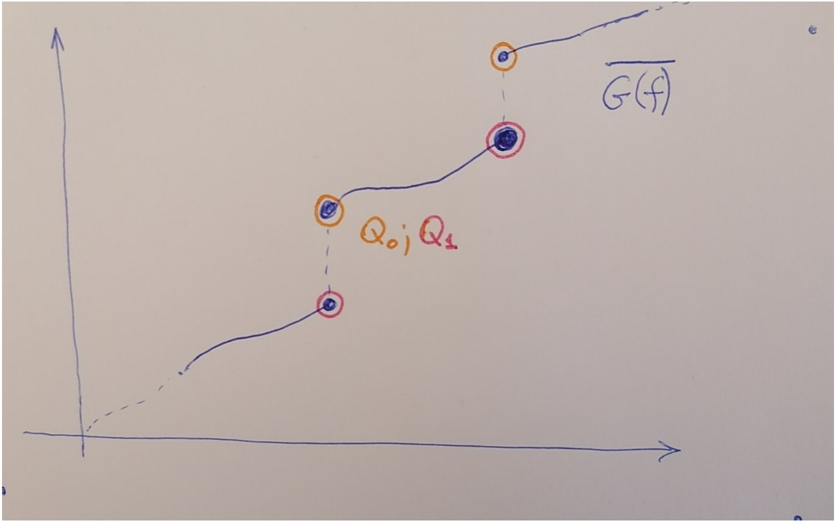

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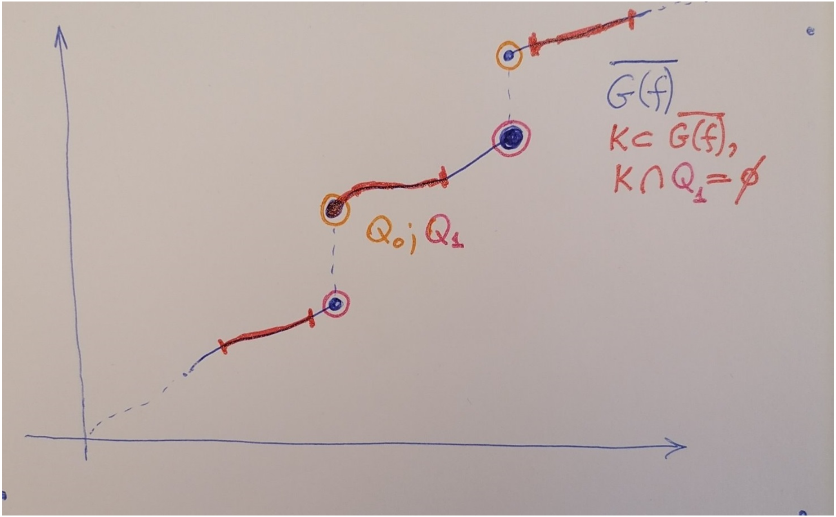

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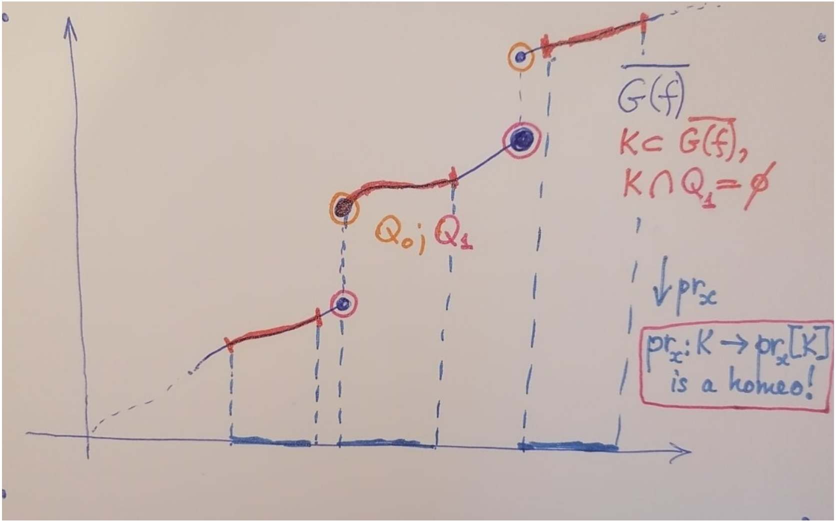

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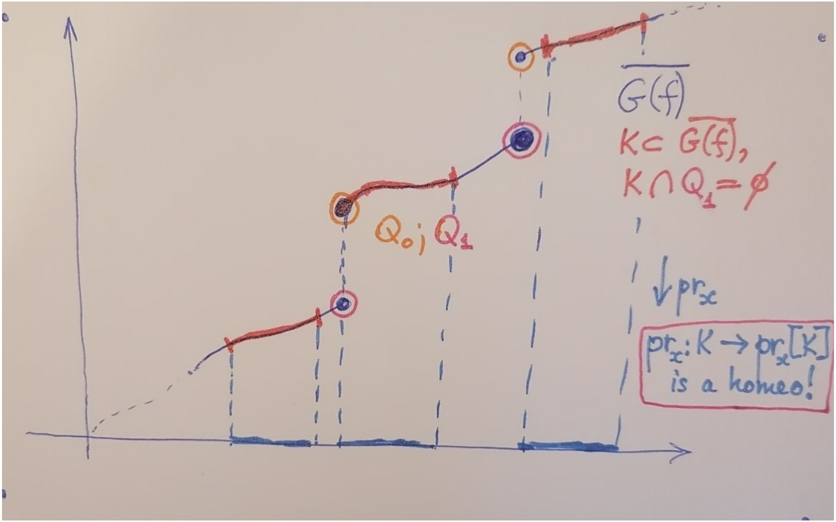

Thus, $\mathrm{pr}_x^{-1} : \mathrm{pr}_x[K] \to K$ is uniformly continuous being a homeomorphism between compacta, and this is f up to countably many "mistakes" on points where f "jumps".

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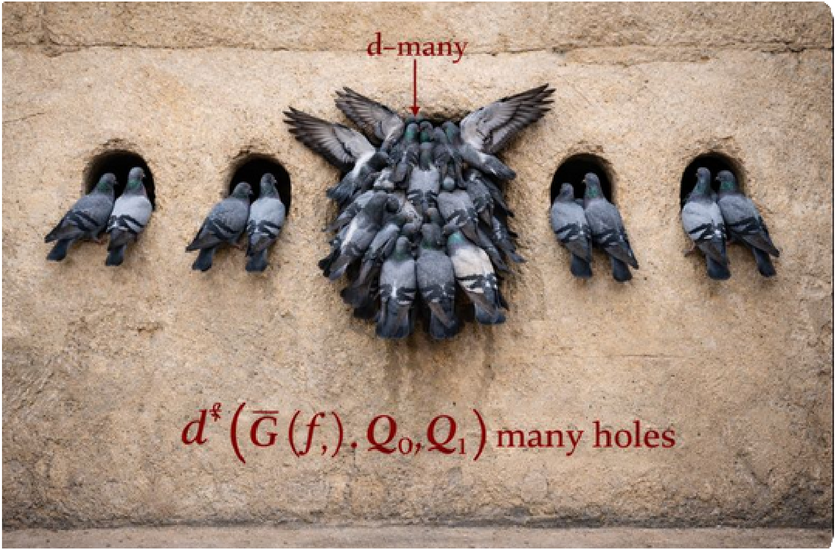
Observation

If $\mathfrak{d}^*(\overline{G(f)}, Q_0, Q_1) < \mathfrak{d}$, then for every $X \subset [0,1]$ of size $\mathfrak{d} = \mathfrak{c}$ and monotone $f : X \to [0,1]$ there exists $Y \in [X]^{\mathfrak{c}}$ such that $f \restriction Y$ is uniformly continuous.

Proof



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The corollary above was proved by Aubrey in 2004.

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- *Pol-Zakrzewski 2024:* Suppose that $\mathfrak{c}$ is singular and there exists $X \subset [0,1]$ of size $\mathfrak{c}$ and continuous $f : X \to [0,1]$ such that $f \restriction Y$ is not uniformly continuous for all $Y \in [X]^{\mathfrak{c}}$.

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The last slide

Thank you for your attention.