Ramsey theoretic notions of convergence
Lecture 1 background/history

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Figure: pigeons and pigeon holes

Pigeon-hole Principle

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Ramsey's Theorem to Nash-Williams

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Nash-Williams's Theorem

For any *barrier* $B \subseteq [\omega]^{<\omega}$, $n \in \omega$ and $f : B \to n$, there is M infinite such that f is constant on $B|_M$.

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So, (1) is the property that every infinite A has an initial segment that is in B

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   B satisfies $B[n] = [\omega \setminus n+1]^n$

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Corollary (Nash-Williams Theorem)

For any barrier B and partition $f : B \to n$ there is an infinite M such that f is constant on $B|_M$.

Rank of a barrier

Definition

For any barrier $B \subseteq [\omega]^\omega$ the tree $T_B \subseteq \omega^{<\omega}$ is defined by first identifying any $s \in [\omega]^{<\omega}$ with its increasing enumeration and letting

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And the rank of the barrier B is $rk(B) = rk(T_B)$

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6. The lexicographical rank of B is $\omega^{\text{rk}(B)}$

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And, of course, X is sequentially compact if every sequence in X has a convergent subsequence.

[1] Bojanczyk, Kopczynski, Torunczyk, *Ramsey's theorem for colors from a metric space*, Semigroup Forum 85 (2012) 182-184.
[2] Banakh, Dimitrova, Gutik. *The Rees-Suschkewitsch theorem for simple topological semigroups*, Mat. Stud., 31(2) (2009) 211-218,

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This is related to Wallace's Theorem, that a compact cancellative topological semigroup is a topological group and the question whether the same holds true if compactness is weakened to countable compactness

Observation: If X is either 2-sequentially compact, or doubly p-compact for some p, then

(∗) for all $f : [\omega]^2 \to X$ there is $x \in X$ so that for all $U \ni x$ open, there are distinct $k, m, n \in \omega$ such that $[\{k, m, n\}]^2 \subseteq f^{-1}(U)$

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EXERCISE: If $x \neq x^2$ then $\exists U \ni x$ open such that $U \cap U^2 = \emptyset$.

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Fix $x \in S$, and define $f : [\omega]^2 \to S$ by $f(\{m, n\}) = x^{n-m}$.
By $(*)$ fix x such that

$\forall U \ni x$ open, there is M of size 3 such that $[M]^2 \subseteq f^{-1}(U)$

CLAIM: x is an idempotent.
EXERCISE: If $x \neq x^2$ then $\exists U \ni x$ open such that $U \cap U^2 = \emptyset$.
Let $k < m < n$ be such that $x^{n-m}, x^{m-k}, x^{n-k} \in U$

Observation: If X is either 2-sequentially compact, or doubly p-compact for some p, then

($*$) for all $f : [\omega]^2 \to X$ there is $x \in X$ so that for all $U \ni x$ open, there are distinct $k, m, n \in \omega$ such that $[\{k, m, n\}]^2 \subseteq f^{-1}(U)$

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But then $x^{n-k} = x^{n-m}x^{m-k} \in U^2$. □

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Definition (Knaust)

A point x in a Hausdorff space X is Ramsey point if for any double indexed sequence $(x_{i,j})_{i,j\in\omega}$, such that $\lim_i(\lim_j(x_{i,j})) = x$, there is an infinite M such that for each neighborhood of x, there is n such that $\{x_{i,j} : i, j \in M \setminus n\} \subseteq U$.

Ramsey notions of convergence

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EXAMPLE: The limit point in the Fréchet-Urysohn fan S_ω is not a Ramsey point.

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The class of angelic spaces with the Ramsey property is countably productive.

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(2) (Knaust) $C_p(X)$ is Ramsey if X is "quasi-Suslin." What if X is "web-compact"?

Discrete images in Hausdorff spaces

[5] T. J. Carlson, N. Hindman, and D. Strauss. *Discrete n-tuples in Hausdorff spaces*, Fundam. Math., 187(2) (2005) 111-126.

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$$R_n(M) = \{A \subseteq [M]^n : \forall N \in [M]^\omega, [N]^n \cap A \neq \emptyset\}$$

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And x is an $R_n(M)$-limit point of $f : [M]^n \to X$ if $f^{-1}(U) \in R_n(M)$ for all nbhds U of x.

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Theorem

If B is the Schreier barrier, there is a linearly ordered space X and $f : B \to X$ such that $f(B|_M)$ is not discrete for any $M \in [\omega]^\omega$.

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For background on barriers see

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