Ramsey theoretic notions of convergence Lecture 2 - Higher Dimensional Sequential Compactness

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Convergence on Barriers

Recall Knaust's notion of a Ramsey point in his study of angelic spaces:

Definition (Knaust)

$x \in X$ is a Ramsey point if given a 2-dimensional array $(x_{i,j})$ in X such that $\lim_i(\lim_j(x_{i,j}) = x$,

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And the class of sequentially compact spaces of Bojańczyk, Kopczyński and Toruńczyk applied to find idempotents in topological semigroups:

Definition (BKT)

X is 2-sequentially compact if for every $f : [\omega]^2 \to X$ there is $x \in X$ and M infinite such that for every open U containing x there is n such that $[M \setminus n]^2 \subseteq f^{-1}(U)$.

n-sequential compactness

The notion of convergence implicit in both previous definitions:

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For $M \subseteq \omega$ infinite, $f : [M]^n \to X$ converges to x if for every neighborhood U of x there is a $k \in \omega$ such that $[M \setminus k]^n \subseteq f^{-1}(U)$.

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And X is n-sequentially compact if every $f : [\omega]^n \to X$ there is $x \in X$ and $M \subseteq \omega$ infinite such that $f \restriction [M]^n$ converges to x.

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Compact metrizable spaces are n-sequentially compact for all n.

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Topological version of Ramsey's Theorem:

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The properties are progressively stronger:

Proposition

If $k \leq n$ and X is n-sequentially compact, then X is k sequentially compact.

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And for B a barrier on ω, X is B-sequentially compact if every $f : B \to X$ there is $x \in X$ and $M \subseteq \omega$ infinite such that $f \restriction (B|_M)$ converges to x.

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Nash-Williams Theorem is equivalent to

Theorem

Metrizable compact spaces are B-sequentially compact for all barriers B.

Remark

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(2) If α is a limit, then there is $\{\alpha_n : n \in \omega\}$ increasing and cofinal in α such that for all $n \in M$, $B[n]$ is uniform of rank α_n.

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REMARKS:
(1) If B is a uniform barrier of rank α then for all infinite M, $B|_M$ is also uniform of rank α.
(2) Every barrier has a restriction on which it is uniform.

A Preorder on Barriers

Theorem

If C is a *uniform barrier* of rank α and B is any barrier of rank $\leq \alpha$ and if X is C-sequentially compact then X is B-sequentially compact.

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Theorem (Todorčević?)

If B and C are any barriers, then there is an infinite M such that either $B|_M \sqsubseteq C|_M$ or $C|_M \sqsubseteq B|_M$.

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Remark: If B and C are any barriers on M and $s \in B|_M$ there is $t \in C|_M$ such that $s \sqsubseteq t$ or $t \sqsubseteq s$

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Hence, if B is of uniform rank α, and if X is B-sequentially compact then X is C-sequentially compact for any barrier of rank $< \alpha$.

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However, it seems analysis of $\sqsubseteq$ is not enough to prove that

Difficult case

If B and C are uniform barriers and $\mathrm{rk}(B) = \mathrm{rk}(C)$ then X is C-sequentially compact if and only if X is B-sequentially compact.

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For example, consider variations of the Schreier barrier:

$$B_f = \{s \in [\omega]^{<\omega} : |s| = f(min(s))\} \text{ } (f \in \omega^\omega \text{ strictly increasing}).$$

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and note that if $f \leq g$ then $B_f \sqsubseteq B_g$

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Definition

Given two barriers B and C on $[\omega]^\omega$, we write $B \preceq C$ if there is a finite-to-one, non-decreasing function $f : \omega \to \omega$ such that for all $M \in [\omega]^\omega$ there is $N \in [M]^\omega$ such that $f \restriction N$ is one-to-one and

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Note that if $B \sqsubseteq C$ then $B \preceq C$ is witnessed by the identity function.

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If $B \preceq C$ and X is C-seq compact, then X is B-seq compact.

α-Sequential Compactness

Definition

X is α-sequentially compact if it is B-sequentially compact for some barrier of uniform rank α. Equivalently, for all barriers of rank $\leq \alpha$. And X is ω_1-sequentially compact if it is α-sequentially compact for all $\alpha < \omega_1$.

Some examples

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For B a barrier on ω the following are ideals on B

1. $\mathrm{FIN}^B = \{A \subseteq B : \{n : A[n] \notin \mathrm{FIN}^{B[n]}\} \text{ is finite}\}$
2. $G_c(B) = \{A \subseteq B \text{ such that } B|_M \not\subseteq A \text{ for any } M\}$

Definition

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Proposition

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Corollary

Rosenthal compacta are ω_1-sequentially compact.

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Definition (Knaust-Todorčević-Corral)

Given a space X and a point $x \in X$, x is a B-Ramsey point in X if for all $f : B \to X$ such that $B \setminus f^{-1}(U) \in FIN^B$ for every neighborhood U of x, there is M infinite such that $f \restriction (B|_M)$ converges to x.

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Points of character $< \mathfrak{b}$ and countably bisequential points are B-Ramsey points.

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Exercises

$\mathfrak{s} = \mathfrak{par}_1 \leq \mathfrak{par}_2 \leq \ldots \leq \mathfrak{par}_B \leq \mathfrak{par}_C \ldots$
(whenever $B \sqsubseteq C$ are of infinite rank)

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Theorem

$\mathfrak{par}_B$ is the minimal cardinal κ such that 2^κ is not B-sequentially compact

So if $\mathfrak{b} < \mathfrak{s}$ we have another example of a sequentially compact not 2-sequentially compact space.

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